Duality between d and $\mathcal{L}$ (I)

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Summary

- Types of Derivatives on M
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- Chain Complexes, Chain Maps, and Homology
- Cochain Complexes, and Cohomology
- Category Theory Perspective

Types of Derivatives on M (I)

- A quick digression on the types of derivs we encounter in diff geom.
- We've encountered several derivation-type objects:
  - Exterior Derivative d. Maps k-forms to $k+1$ forms. In fact, d is the primary reason we care about forms (as opposed to general tensors).
  - Lie Derivative $\mathcal{L}_V$. Maps (j,k)-tensor fields to (j,k)-tensor fields. It is how a tensor field flows along v.f. V.
  - Covariant Derivative ∇. Operates on tensor fields, but the notation hides its dependence on a choice of connection (including torsion).
  - Interior Product i_X. Though not usually thought of as a derivative, it behaves very much like one in a number of senses. We'll defer this.

Types of Derivatives on M (II)

- d is "natural," a manifold-level object which arises from the canonical de Rham complex and requires no additional info.
- $\mathcal{L}$ is "natural," in the sense that it arises from the v.f. Lie Bracket, a manifold-level object. Regarded as a map $\Gamma(TM) \times \Gamma(T^j_k M) \to \Gamma(T^j_k M)$ it is canonical to the manifold, but any given application requires a specific $V \in \Gamma(TM)$.
- Covariant derivs are not considered "natural." Why not do the same as for $\mathcal{L}$ and consider maps of spaces? The distinction is somewhat semantic, but driven by a few major considerations:
  - $\mathcal{L}$ relies on the canonical lie bracket and needs no additional defs.
  - Connections are more complicated objects.
  - In principle, not every M has a connection definable on it.
  - Unlike d and $\mathcal{L}$, the natural domain for conns/torsions is P not M.

Types of Derivatives on M (III)

- Recall two canonical structures which exist on any smooth manifold:
- The de Rham complex, defining a graded exterior algebra of forms along with an exterior derivative. It is the unique extension (subject to certain simple conditions) from a simple 0 and 1-form definition to the full graded algebra. We build it inductively (on each chart) from a free algebra generated by symbols $dx_1 \ldots dx_n$ quotiented by $dx_i dx_j = -dx_j dx_i$, using smooth functions as coefficients.
- The vector field commutator LA, given by $[v, w](f) \equiv v(w(f)) - w(v(f))$. This also is known as the Lie Derivative (or, more precisely, the LA of v.f.s is isomorphic to the LA of infinitesimal generators of flows, representing the two views of v.f.s). The Lie Derivative approach can be extended to tensor fields. So can the regular v.f. LA but the interpretation is a bit less intuitive. It's easier to think of sliding tensors along a v.f..
- Let's review the defs more precisely.

De Rham Complex on M

- Formally, the de Rham complex is defined as (ala Bott & Tu):
  - Consider chart $U \subset M$, with local coords $x_1 \dots x_n$.
  - At $m \in M$, define graded algebra Ω^* over $\mathbb{R}$ as generated by symbols $dx_i, dx_i dx_j, dx_i dx_j dx_k, \dots$ modulo $dx_i dx_j = -dx_j dx_i$.
  - The algebra of diff forms is $\Omega^*(U) = \{C^\infty \text{ fns on U }\} \otimes_{\mathbb{R}} \Omega^*$. I.e., we replace real coeffs with smooth fns on U.
  - Each p-form can be uniquely written $\omega = \sum f_I dx_I$ where I denotes $\{i_1 < i_2 \cdots < i_p\}$ and dx_I denotes $dx_{i_1} \dots dx_{i_p}$, and f_I are smooth fns.
  - 0-forms are defined as smooth fns $U \to \mathbb{R}$
  - $\wedge : \Omega^* \times \Omega^* \to \Omega^*$ takes a p and q form into a $(p+q)$ form via $(\sum f_I dx_I) \wedge (\sum g_J dx_J) = \sum f_I g_J dx_I dx_J$, which can be written as $\sum h_K dx_K$ once we permute everything into the right order. The wedge product obeys $\mu \wedge \nu = (-1)^{pq} \nu \wedge \mu$.
  - $d : \Omega^* \to \Omega^*$ takes p-forms to $(p+1)$-forms, and is defined via
    - $df = \sum \frac{\partial f}{\partial x_i} dx_i$
    - $d(\sum_I f_I dx_I) = \sum_I df_I dx_I$
    - $d(\mu \wedge \nu) = (d\mu) \wedge \nu + (-1)^p \mu \wedge (d\nu)$.
  - We stitch the per-chart complexes together via a partition of unity.

Review of a few Diff Geom Facts

- We'll denote by $A()$ an operator which enforces antisymmetry amongst the indices of its arg. It is idempotent.
- $d^2 = 0$.
- Closed form: $d\omega = 0$
- Exact form: $\omega = d\nu$ for some form ν. All exact forms are closed.
- $df(V) = V(f)$. This follows from the def of d.
- $i_X : \Omega^* \to \Omega^*$ takes p-forms to $(p-1)$ forms
  - $(i_X\omega)(X_1 \ldots X_n) = A(\omega(X, X_1 \ldots X_n))$.
  - I.e., we contract against X, antisymmetrizing appropriately.
  - Although only defined for $p > 0$, it is convenient to define $i_X f = 0$ for fns (i.e. 0-forms).
- Contraction of (j, k)-tensor (or tensor-field):
  - $\exists$ bilinear form $T_xM \times T_x^*M \to \mathbb{R}$ defined by $(\omega, V) \to \omega(V)$. This extends to v.f.s. In any basis it induces a dual basis.
  - We can contract an upper and lower index of T, but not two uppers or two lowers (at least not without an extra inner-product).
  - In any basis for V, with induced tensor (and dual) basis for T, we just sum $T^{\cdots i \cdots}_{\cdots i \cdots}$ in the relevant indices (like a trace).
  - We'll denote contracting i^{th} upper with j^{th} lower index $C^i_j(T)$.

Lie Derivative and Bracket

- Recall the Lie Bracket on M:
  - Canonical LA of v.f.s
  - $([V, W])(f) \equiv V(W(f)) - W(V(f))$.
- $\mathcal{L}_V$ is defined axiomatically as follows:
  - Usual deriv on fns: $\mathcal{L}_X f = X(f)$.
  - Action on v.f.s: $\mathcal{L}_X Y = [X, Y]$
  - Leibnitz: $\mathcal{L}_X(T_1 \otimes T_2) = (\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_X T_2)$
  - Commutes with d: $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$ on forms.
  - Commutes with contraction: $\mathcal{L}_X(C^i_j(T)) = C^i_j(\mathcal{L}_X T)$ for any contraction of specific upper and lower index.
- The general formula is $(\mathcal{L}_X T)(\omega_1 \ldots \omega_n, v^1 \ldots v^m) = X(T(\ldots)) - \sum_{i=1}^n T(\ldots \mathcal{L}_X \omega_i \ldots) + \sum_{i=1}^m T(\ldots \mathcal{L}_X v^i \ldots)$. I.e, we have an overall term, then we sum over application to each argument with $-$ for 1-forms and $+$ for v.f.s.

Chain Complexes (I)

- Chain Complex: a sequence of abelian groups C_i with maps $d_i : C_i \to C_{i-1}$, s.t. $d_i \circ d_{i+1} = 0$.
- Use d collectively for d_i. Clear from domain which to use.
- Cycles: Elements of $\ker d_i \subset C_i$
- Boundaries: Elements of $\operatorname{Im} d_{i+1} \subset C_i$
- Cycle condition: $d_i \circ d_{i+1} = 0$.
- I.e., a chain complex is a sequence of abelian groups, and homomorphisms between them that obey the cycle condition.
- Written $\cdots \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \cdots$, where d is understood to be the relevant d_i at each step. The cycle condition is denoted $d^2 = 0$.
- Sometimes written facing left instead. The key feature is that the arrow points in order of descending n. The n has semantic significance for each application. Usually dimension or degree, etc. We'll discuss more shortly.
- Denote a chain complex C, which includes both the C_i and the d_i.

Chain Complexes (II)

- Ex. Boundary map for a simplicial, singular, or CW complex of a topological space X. We'll consider cells.
- The relevant n is cell dimension.
- The relevant C_i is a v.s. with basis the cells of dimension i.
- The relevant d is the boundary map attaching an n-cell to its boundary $(n-1)$-cells. In this case, the coefficient field could be Z_2, but fields with nonzero characteristic are troublesome. We typically choose Z (making C_i a module, not a v.s.). Later we'll upgrade the coeffs to $\mathbb{R}$, making it a v.s. again.
- Cycles: A combination of cells that has no boundary.
- Boundary: A combination of cells which is the boundary of some combination of cells of higher dimension.
- d is denoted ∂, and $\partial^2 = 0$ is the cycle condition.
- This makes sense since the boundary of a cell is closed, and has no boundary of its own. Ex. a 2-cell closed disk with boundary an oriented circle.
- This actually is where the term "cycle" comes from.

Chain Maps and Homology

- Chain Map; Given two chain complexes C and C', a seq of abelian-group-homomorphisms $\alpha_i : C_i \to C'_i$ s.t. $d'_i \circ \alpha_i = \alpha_{i-1} \circ d_i$.
- I.e., each square commutes.
- Denote $\alpha : C \to C'$, and its condition as $d \circ \alpha = \alpha \circ d$, with the above meaning intended.
- Homology: Given chain complex C, a seq of abelian groups $Hom_i(C) \equiv \ker d_i / \mathrm{Im}\ d_{i+1}$.
- I.e., the cycles modulo the trivial cycles, those which are 0'd out due to the cycle condition.
- Ex. for a cell complex boundaries of cells have no boundaries of their own and thus are trivially cycles.
- A chain map induces a map of homologies ($H_k \to H'_k$). This is obtained by mapping elements directly. The result survives the quotient map and is well-defined (i.e. independent of trivial cycles) in terms of homology classes.
- SES of chain complexes gives rise to LES of homology groups. There is an extra map involved which goes from $H_n(C'') \to H_{n-1}(C)$, and obtained from zig-zag lemma.

Cochain Complexes and Cohomology (I)

- A cochain complex goes in the opposite direction.
- Seq of abel groups C^i with maps $d^i : C^i \to C^{i+1}$, s.t. $d^{i+1} \circ d^i = 0$.
- Written $\cdots \leftarrow C^n \overset{d}{\leftarrow} C^{n-1} \overset{d}{\leftarrow} C^{n-2} \overset{d}{\leftarrow} \cdots$, where d is understood to be the relevant d^i at each step. The cocycle condition usually is denoted $d^2 = 0$, not to be confused with the map $d^2 : C^2 \to C^3$.
- Once again, can be drawn left or right, but key is that arrows ascend.
- Cochains are nothing more than chains with a particular labeling convention. As we will discuss, there may be specific reasons for referring to particular chains as chains and others as cochains. The category of chains is self-dual, in the sense that every chain has an associated dual chain (often called its cochain). However, given a particular object, the choice of whether to call it a chain or cochain is a (strongly motivated) convention.
- What is termed chain vs cochain and homology vs cohomology does follow a certain logic in practice, but at some level everything is just chains and homology!
- Ex. if we have reason to refer to something as a chain (rather than cochain), and its dual also is of interest, we would term that dual a cochain — and vice versa if starting with something we term a cochain.

Cochain Complexes and Cohomology (II)

- But suppose we start with some system and a chain complex arises from it. How do we decide whether to refer to this as a chain or cochain. Does it matter how we label it?
  - The sequence may start or end. Usually this happens at something called index 0 (or maybe -1). If it ends, we must be descending, and it is natural to speak of a chain. If it starts, we must be ascending and it is natural to speak of a cochain. If the seq continues in both directions to infinity (or is finite), then either makes sense.
  - There often is a semantic meaning to the index. Ex. if the index is a dimension or degree then a given seq naturally would be called a chain if descending or cochain if ascending.
  - If the chains arise from objects (ex. top spaces), then the functors which produce them may be covariant or contravariant, depending on how they affect chain maps. In that case we usually would speak of (respectively) chains and cochains.
- Note that there is no nontermination requirement for chains or cochains. They can be finite (starting and ending in a trivial 0 group) or infinite on one side or both.
- Mathematically, a cochain just a chain with different labels. This will be important.

Cochain Complexes and Cohomology (III)

- Elements of $(\ker d^i) \subset C^i$ are cocycles or "closed."
- Elements of $(\mathrm{Im}\ d^{i-1}) \subset C^i$ are coboundaries or "exact."
- $d^{i+1} \circ d^i = 0$ (i.e. $d^2 = 0$) is known as the "cocycle" condition.
- Cohomology: The cohomology groups of a cochain are the same as the homology groups of it viewed as a chain.
- H^n is used for cohomology groups rather than H_n.
- $H_n = \ker d_n / \mathrm{Im}\ d_{n+1}$ and $H^n = \ker d^n / \mathrm{Im}\ d^{n-1}$. This is consistent with the directionality convention. Arrows always start at C_n or C^n.
- There is no difference between cohomology and homology ab initio. The differences arise when we define functors — or deal with the cohomology of duals to chains.
- In the latter case, it is meaningful to speak of the homology AND cohomology of chains (or of the objects from which they arise, such as top spaces).
- Let's look at the dual construction a bit more closely.

Cochain Complexes and Cohomology (IV)

- Note: the dual of a chain doesn't just have reversed arrows.
- Let's construct the dual of a mathematical chain. I.e., what we term the chain of given cochain, or the cochain of a given chain.
- The dual of an abelian group is the group of homomorphisms $G \to Z$.
- A finitely generated group is self-dual, so $C^i \approx C_i$.
- Given $d : C_i \to C_{i-1}$, the dual $d^* : C^{i-1} \to C^i$ is defined by $d^*(x) = x \circ d$ for $x \in C^{i-1}$. Note that $d^*(x) \in C^i$ and thus is a homomorphism $C_i \to Z$. On the other hand, $x \circ d$ takes an element of C_i, produces an element of C_{i-1}, and maps it to Z. So the two are compatible (we haven't proven they are equal, though).
- Note: d^* is shifted by one. For both chains and cochains we label the d's relative to their domain, so d_i is dual to $d^{*(i-1)}$.

Cochain Complexes and Cohomology (V)

- Back to cochains and cohomology, we already have an example of a cochain complex.
- The de Rham complex defines a mathematical chain.
- Because the degree (i.e. grade) is a natural index, and the seq has a start, we term it a cochain complex.
- Equally important, the generating functor can be seen to be contravariant. We won't show that here, though.
- $C^k = \Omega^k$, the k-form grade in the graded algebra. Viewed at a point, the coefficients are real, so it naturally is a real v.s. with basis the dx_I of degree k.
- d, now denoting the exterior derivative, maps C^k to C^{k+1}. It satisfies the cocycle condition.
- We thus have a cochain complex.
- The cohomology groups are just $H^n = \ker\, d^k / \mathrm{Im}\, d^{k-1}$, with d^k denoting the action of d restricted to Ω^k.
- They correspond to the quotient of closed k-forms/exact $(k-1)$-forms.
- The exact forms are trivially closed, so this amounts to counting the nontrivial closed forms.

Categories (I)

- Chain complexes and Chain maps form a category CH. It is self-dual as a cat, but this does not mean each object is self-dual. Chains and cochains are dual pairs of objects.
- Sometimes people speak of subcategories of CH: chains bounded below (i.e. with start), chains bounded above (i.e. with end), or finite chains. The latter is self-dual as a cat, while the first two are dual to one another (corresponding to cats of cochains and chains respectively). We'll work in CH, but it is important to keep in mind what one is talking about.
- Each Homology group H_n is a covar functor CH $\rightarrow$ ABEL.
- Chain or cochain complexes typically arise from objects is some cat MYCAT, via some functor $F : MYCAT \rightarrow CH$. Ex. top spaces $\rightarrow$ CW complexes or smooth manifolds $\rightarrow$ de Rham complexes.
- We define the "homology of MYCAT" as the functors $H_n \circ F$.
- Cohomol on cochains is just homol on them as mathematical chains.
- The functor CH $\rightarrow$ ABEL remains the homology functor, and thus covariant. Whether or not we call the maps between chains "chain maps" or "cochain maps", it does not flip them.

Categories (II)

- We speak of the relevant $H_n \circ F$ as either the homol or cohomol of MYCAT, according to whether we term the mathematical chains which F produces "chains" or "cochains."
- So why do we say that cohomology is a contravariant functor?
- The reason depends how the cohomology in question arises.
- First, suppose it arises via the dual construction. I.e. we start with chains and homology (ex. cell-complex chains), but also care about the corresponding dual construction.
- Let D be the dual functor on CH. It takes chains and chain-maps to their duals. But it is a contravar functor CH$\rightarrow$ CH because it flips the chain maps in the process.
- The relevant functors then are $H^n = H_n \circ D$, because we are binding outselves to duals of existing objects.
- If the chains and homology arose from MYCAT via a (covar) functor F, then $H_n \circ D \circ F$ are the cohomology groups. Again, they are contravariant.

Categories (III)

- Next, suppose we start with a natural chain construction. I.e., a functor F:MYCAT $\to$ CH as before. It takes objects in MYCAT to mathematical chains, and arrows in MYCAT to chain maps.
- Now suppose this functor happens to be contravar, flipping arrows.
- Ex. the de Rham complex w/ MYCAT= DIFF (smooth manifolds).
- In that case, we call the mathematical chains it produces "cochains."
- MYCAT's "cohomology" functors are $H_n \circ F$, where H_n are the usual homology functors.
- These are contravariant functors (contravar $\circ$ covar $=$ contravar).
- When we produce mathematical chains from MYCAT, this is the main criterion for whether we have chains/homology or cochains/cohomology. If F is covar it is a homology theory, and if F is contravar we have a cohomology theory.
- This plays well with the start/end and dim/degree criteria we discussed earlier.
- Note that both generative mechanisms produce a "cohomology" theory, effected through contravar functors. They differ only in how we choose our cochain complex: ab initio (ex. de Rham) or dual construction (simplicial Cohomology).