PARTITIONS OF SETS

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Contents

1. Partitions of a Set

1.1. **Basic Definitions.** A **partition of set** S is a set of pairwise-disjoint nonempty subsets of S whose union is S itself. We'll denote by $Par(S)$ the set of partitions of S and by $CPar(S)$ the set of countable partitions of S. The elements of a partition will sometimes be referred to as **classes**.

We'll denote the powerset of S by 2^S rather than $P(S)$, to avoid notational confusion.

For $CPar(S)$, "countability" refers to the number of classes in the partition, not the sizes of those classes.

We'll typically leave off the "of S" moniker when the relevant S is clear.

"Pairwise-disjoint" means $s_i \cap s_j = \emptyset$ for all i, j, whereas "mutually-disjoint" means $\cap_i s_i = \emptyset$. Pairwise-disjoint implies mutually disjoint, but not the converse. Mutually-disjoint is a much weaker condition. Even a single $s_i \cap s_j = \emptyset$ implies it. For finite intersections, mutually-disjoint is, in fact, equivalent to the existence of a disjoint pair. However, for infinite intersections it can hold even when no such pair exists. [Ex. let $s_n = (0, 1/n)$ for $n > 0$. For any $x \in (0,1)$. $\exists n$ s.t. $1/n < x$, so $x \notin (0, 1/n)$ and thus $x \notin \cap s_i$. Since no x is in it, $\cap s_i = \emptyset$. However, any $s_n \cap s_m \neq \emptyset$, because $x = \frac{1}{2}\min(1/n, 1/m)$ is in it.] Note that neither pairwise-disjoint nor mutually-disjoint is confined to countable collections of sets. The collections may be arbitrary in size.

We'll refer to the partition $\{S\}$ as the **trivial partition** and the partition $\{\{x\}; x \in S\}$ (i.e. all single-element sets) as the **singleton partition**.

The partitions of S are in one-to-one correspondence with the equivalence relations on S. They define one another in the obvious way.

For partition B, $x \sim y$ iff x and y are in the same class $b \in B$.

A partition B is a **refinement** of partition B' (on the same S) if every $b \in B$ is $\subseteq$ some $b' \in B'$. Put another way, the classes of B' are unions of classes of B. We say a refinement is **proper** if $B \neq B'$.

Conversely, we'll say that B' is a "coarsening" of B. This is non-standard terminology.

Note that there is no useful notion of "sub-partition", only refinement. If B is a partition, no proper subset (or proper superset) of it can be a partition. If we remove a class, the union is no longer S. If we add a class, we lose disjointness.

In these notes, we'll assume the axiom of choice.

We can **restrict a partition** B of S to subset $S' \subset S$ via $B' \equiv \{b \cap S'; b \in B\}$ (excluding any empty intersections). B' is a partition because it remains disjoint and covers S'. We'll say that S' is **compatible with** B iff $B' \subseteq B$. This means that classes of B are eliminated but not truncated when we restrict to S'.

Clearly, $B' = B$ iff $S' = S$.

Going the other way, if we have B of S, we may consider how to extend it to a partition B' of $S' \supset S$. There are many ways to do this, because we can attach the new elements to any existing partitions or form new classes from them. If we also require that S' be compatible with the extended partition, we can only form new classes from the new elements. However, there still may remain many ways to do this (if we're adding more than one new element).

1.2. **Equivalence relations and quotients.** As mentioned, there is a natural bijection between partitions of S and equivalence relations on S. Suppose we have an equivalence relation $\sim$ on S, with corresponding partition B. I.e., $s \sim s'$ iff s and s' are in the same class of B.

We can form the quotient $S/{\sim}$ to get the set of equivalence classes, which is naturally bijective with B.

Suppose we have equivalence relations $\sim$ and $\sim'$ on S. We can't compute $(S/{\sim})/{\sim'}$ unless $\sim'$ respects the classes of $\sim$. I.e., $\sim'$ must be a coarsening of $\sim$. In this case, $(S/{\sim})/{\sim'}$ is even smaller (i.e. corresponds to fewer, coarser classes on S).

If B and B' are partitions of S corresponding to $\sim$ and $\sim'$, then $(S/\sim)/\sim'$ makes sense iff B is a refinement of B'. $S/\sim$ can be thought of as coarsening the singleton partition to B, and $(S/\sim)/\sim'$ can be thought of as coarsening B to B'.

1.3. **Induced Power-Set Map.** It is often useful to treat $s \subseteq S$ as both a subset of S and an element of 2^S. Where the usage is not obvious from the context (or when we wish to emphasize it), we'll write s for the former and $\widehat{s}$ for the latter. On the rare occasions when we need to go the other way, we'll write $\breve{x}$. We'll stick with $\subseteq$ for the subset partial order on 2^S, although $\leq$ would perhaps be more appropriate.

I.e., if $s \subseteq S$ viewed as a subset, then $\hat{s}$ is s viewed as an element of 2^S, and if x is an element of 2^S, then $\breve{x}$ is x viewed as a subset of S.

Our distinction between s and $\hat{s}$ (and x and $\breve{x}$) is pedantic and not anything profound. We introduce it only to avoid notational confusion in places and to relieve the reader of some of the burden of carefully tracking the type of each object in every statement. We won't be religious about its use, though, so readers are advised to perform their own typechecking if the usage seems unclear.

In keeping with ordinary set-theory, if $f : S \to S'$, we'll write $f(x)$ for the image of element $x \in S$ and $f(s)$ for the image of set $s \subseteq S$, viewed as a subset of S'. If we wish to view the image as an element of $2^{S'}$, we will write $\widehat{f(s)}$.

We'll denote by $Map(S, S')$ the set of all maps between S and S', and by $Iso(S, S')$ the set of all bijective maps. We'll use $Aut(S)$ for the set of all bijective maps from S to itself.

$Map(U, V)$ is ordinarily denoted V^U, but we wish to avoid relying too much on potentially overlapping syntactic sugar. Note that $Aut(S) = Iso(S, S)$. We use $Iso(S, S')$ for bijections because these are the isomorphisms of sets. We'll later define analogous $Iso()$ sets for other types of objects.

Given sets S and S', any map $f : S \to S'$ induces a map $f' : 2^S \to 2^{S'}$.

This is obtained via $f'(\hat{s}) \equiv \widehat{f(s)}$. I.e., we map each set to its image, with both now viewed as elements of the corresponding power-sets.

The converse does not hold. $Map(S, S')$ is far smaller than $Map(2^S, 2^{S'})$. Only a tiny fraction of the latter maps are induced by the former. However, no two maps $f_1, f_2 : S \to S'$ induce the same power-set map.

The map taking each f to the induced f' can be written $\gamma : Map(S, S') \to Map(2^S, 2^{S'})$. This γ is injective but (highly) non-surjective. This non-surjectivity of γ persists even if we restrict it to bijections $\gamma|_{Iso(S,S')}$. As we'll see in a moment, f is bijective iff the induced f' is bijective — so $\gamma|_{Iso(S,S')}$ is a map $Iso(S, S') \to Iso(2^S, 2^{S'})$. As a restriction, it remains injective; however, it too is highly non-surjective.

Note that we *cannot* construct a natural map $\beta : Map(2^S, 2^{S'}) \to Map(S, S')$. It may be tempting to try to define $\beta(g)$ by the behavior of g on singleton sets. However, a general map $g \in Map(2^S, 2^{S'})$ need not take singleton sets to singleton sets. Unless it does so, we won't have an induced map $S \to S'$. Suppose $|S| = 5$ and $|S'| = 4$. Then $|Map(S, S')| = 4^5 = 2^{10}$. $|2^S| = 2^5 = 32$ and $|2^{S'}| = 2^4 = 16$, so $|Map(2^S, 2^{S'})| = 16^{32} = 2^{128}$. Of the elements of 2^S, 27 are non-singleton, and of the elements of $2^{S'}$, 12 are non-singleton. This means that there are $12^{27} \cdot 2^{10}$ maps that take singletons to singletons, or 12^{27} maps per element of $Map(S, S')$. However, the remaining $2^{128} - 12^{27} \cdot 2^{10}$ maps take at least one singleton to a non-singleton. The fraction of maps that take all singletons to singletons is roughly 0.0000004.

Prop 1.1: (i) f is injective iff the induced $f' : 2^S \to 2^{S'}$ is injective. (ii) f is surjective iff f' is surjective. (iii) f is bijective iff f' is bijective. (iv) If f is bijective, then f^{-1} induces f'^{-1}. (v) If f_1 and f_2 are composable (i.e. $f_1 : S \to S'$ and $f_2 : S' \to S''$), then $(f_2 \circ f_1)' = f_2' \circ f_1'$.

Note that the "iff" *only* holds for the induced f'. As mentioned, $f : S \to S'$ contains far less information than a general map $f' : 2^S \to 2^{S'}$. If we allow f' to be a general map — even a bijection — there may be no corresponding f which induces it. Only a small subset of the maps f' (or even of the bijective maps f') correspond to maps $f : S \to S'$. To see this, suppose that we have some general bijection $f' : 2^S \to 2^{S'}$. Since x is the intersection of all sets containing itself (i.e. $x = \cap_{\{s\in S; x\in s\}} s$), we would need (among other things) that $s \subset s'$ implies $f'(s) \subset f'(s')$ (viewed as subsets of S'), which need not be true of a general bijection f'.

Pf: (i) forward injection: Suppose f is injective. Consider $s, s' \subseteq S$ with $s \neq s'$. There must be some x that is in one but not the other. Let's assume $x \in s$ and $x \notin s'$. Then $f(x) \in f(s)$. Since f is injective, no other y can map to that $f(x)$. Therefore, $f(x) \notin f(s')$, and $f(s) \neq f(s')$, which means $f'(\hat{s}) \neq f'(\hat{s'})$. (i) backward injection: Now, suppose that the induced map f' is injective. For a singleton set $\{x\}$, we have $f(\{x\}) = \{f(x)\}$. I.e., since f maps points to points, it takes a singleton set domain to a singleton set image. This means that the induced f' maps singleton sets to singleton sets. Since f' is injective, if $\{x\} \neq \{y\}$ then $f(\{x\}) \neq f(\{y\})$. This means $\{f(x)\} \neq \{f(y)\}$ and thus $f(x) \neq f(y)$. But $\{x\} \neq \{y\}$ iff $x \neq y$. So f is injective too. (ii) forward surjection: Suppose f is surjective. Pick any nonempty $s' \subseteq S'$. Since f is surjective, $f^{-1}(s') \neq \emptyset$. In general $f(f^{-1}(s')) \subseteq s'$, but for surjective f we have equality. This tells us that $f'(\widehat{f^{-1}(s')}) = \hat{s'}$ for some $f^{-1}(s')$. So f' maps some $\hat{s}$ to $\hat{s'}$ and is surjective. (ii) backward surjection: Suppose f' is surjective. Then $f'^{-1}(\{x'\}) \neq \emptyset$ is defined for every singleton set $\{x'\}$. Although $f^{-1}(x')$ need not be a singleton set itself (since f need not be injective), every element in it must map to x'. We therefore have at least one element of S that maps to x', and f must be surjective. (iii) bijection: The bijective iff follows automatically from (i)-(ii). (iv) inverse induces inverse: Now, let f be bijective. Consider $(f^{-1})'$, the map $2^{S'} \to 2^S$ induced by $f^{-1} : S' \to S$. Since f is bijective, so is f^{-1}, and from (iii) we have that f' and $(f^{-1})'$ are as well. Suppose we start with $s \subset S$. Then $f^{-1}(f(s)) = s$. This means that $(f^{-1})'(\widehat{f(s)}) = \hat{s}$, which tells us that $(f^{-1})'(f'(\hat{s})) = \hat{s}$. I.e., $(f^{-1})' = f'^{-1}$. (v) composition: This is obvious, but let's go through it anyway. Let $s \in S$, and consider $f_2' \circ f_1'$. By definition $f_1'(\hat{s}) = \widehat{f_1(s)}$ and $(f_2 \circ f_1)'(\hat{s}) = \widehat{(f_2 \circ f_1)}(s)$. Similarly, for any s', $f_2'(\hat{s'}) = \widehat{f_2(s')}$. Let $s' = f_1(s)$, so $f_1'(\hat{s}) = \widehat{f_1(s)}$. Then $f_2'(f_1'(\hat{s})) = f_2\widehat{(f_1(s))} = (f_2 \circ f_1)'(\hat{s})$.

For any $f : S \to S'$ and specific partition $B \in Par(S)$, the induced f' restricts to a map $B \to 2^{S'}$. However, the image of this map need not be a partition of S'. I.e., we can have $f'(B) \notin Par(S')$.

If f is not surjective, its image won't cover S', and we never get a partition. Therefore, surjectivity is a requirement. What about injectivity? Suppose f is non-injective and B is nontrivial. Consider distinct classes $b, b' \in B$ (i.e. $b, b' \subset S$). Since f is non-injective, it is possible that $f'(b)$ and $f'(b')$ are not disjoint. If $f(x) = f(x')$ for some $x \in b$ and $x' \in b'$, then the images will not be disjoint. As we will see, even if f is non-injective it is possible that a given B maps to a partition. It depends how much we demand. If we require a partition of S' that looks just like B (i.e. the $f(b)$'s are nonempty, disjoint, and form a partition), then we are highly constrained. This does *not* require that S and S' be bijective, but it does require that f avoid non-injectivity across classes of B. Noninjectivity within each class is fine, though. On the other hand, if we just require f to induce a map from B to some partition B', without the induced map $f'|_B$ being injective, we have a lot more flexibility. In that case, the $f(b)$'s can overlap. There always is *some* B' which works, but it may be the trivial one. The questions of (i) which B''s work and (ii) whether there is a finest such B' will be addressed below. We'll see that the relevant condition for f to induce a partition map between a given B and B' is that (aside from f being surjective) for each $b \in B$, $f(b) \subseteq b'$ for some $b' \in B'$.

We potentially can overcome a lack of injectivity of f by taking a quotient and a lack of surjectivity of f by restricting ourselves to $f(S) \subset S'$. There is no notion of 'kernel' when it comes to maps between sets, but we can use the obvious equivalence relation: $x \sim_f y$ iff $f(x) = f(y)$. Denoting $Q \equiv S/\sim_f$, this induces a bijective map $\tilde{f} : Q \to \text{Im } f$, which means that the induced powerset-map $\tilde{f}' : 2^Q \to 2^{\text{Im } f}$ is also bijective. Although 2^Q is not a subset of 2^S, $2^{\text{Im } f}$ is a subset of $2^{S'}$ (it contains all subsets of S' consisting solely of elements drawn from $\text{Im } f \subset S'$). In effect, we're using $\sim_f$ to define a partition of S and then confining ourselves to partitions which are coarsenings of it. We won't use this, but we will have more to say about the use of equivalence relations and quotients.

For a given $f : S \to S'$, there are various associated maps which come into play when discussing partitions. Notions such as injectivity, surjectivity, and bijectivity can apply to some or all of these, and we must be careful to distinguish exactly which maps we are talking about:

- The map $f : S \to S'$ itself.
- The induced powerset-map $f' : 2^S \to 2^{S'}$, taking each subset of S to a subset of S' but with both viewed as elements of powersets. Restricted to a given partition B of S, $f'|_B$ takes B to some set of subsets of S' (i.e. a subset of $2^{S'}$). This image of B, which we'll usually write as $f'(B)$, may or may not be a partition of S'.
- The induced double-powerset-map $f'' : 2^{2^S} \to 2^{2^{S'}}$. This is the map induced by f', which in turn was induced by f. Restricted to $Par(S)$, it takes each partition of S to a set of subsets of S'. The

latter may or may not be a partition of S' (i.e. an element of $Par(S)$). Overall, $f''(Par(S))$ may or may not be a subset of $Par(S')$.

We can do the same by restricting f'' to $CPar(S)$. $f''(CPar(S))$ may or may not be a subset of $Par(S')$. If it is, it may or may not be a subset of $CPar(S') \subseteq Par(S')$.

- For a given partition B of S, each class $b \in B$ has an induced map $f|_b : b \to f(b)$, which takes it to a subset of S'.

1.4. **Maps between Partitions.** Given partitions B of S and B' of S', we can consider various notions of a "map between partitions".

When speaking of maps, there are four levels of structure involved:

- Point maps $f : S \to S'$.
- Maps between partitions $g : B \to B'$.
- Point maps between classes of a partition $h_b : b \to g(b)$ for each $b \in B$.
- Maps between the set of partitions of S and the set of partitions of S'.

We'll introduce these later, where the relevant sets of partitions will be denoted $Par(S)$ and $Par(S')$.

At the simplest level, we can consider maps between B and B' directly. However, these are specific to the given B and B', and are uninteresting in isolation. Instead, we'll focus on point maps $f : S \to S'$ and how they behave in relation to a choice of B and B'.

We could also formulate things in terms of both a g and a corresponding set $\{h_b; b \in B\}$. We'll discuss later how this combination relates to a choice of f.

This is similar to the approach taken in other areas of math where the object of interest consists of a set along with a set of its subsets. For example, in topology we consider whether a map $f : S \to S'$ is "continuous" or a "homeomorphism" based on its behavior relative to specific topologies on S and S'. Similarly, in measure theory we consider whether $f : S \to S'$ is "measurable" or "bimeasurable" based on its behavior relative to specific σ-algebras on S and S'. In neither case, do we work directly with maps between the topologies (i.e. maps from the set of open sets to the set of open sets) or σ-algebras (i.e. maps from the set of measurable sets to the set of measurable sets) as our fundamental notion. Such maps are induced rather than what we start with.

This distinction is not necessary for objects consisting of an algebraic structure on a set. In that case, we have no choice but to consider set-maps which satisfy certain constraints. It is the presence of two sets in our case (i.e. S and a set of its subsets) which presents a possible alternative.

There is no widespread term for a measurable function with measurable inverse. However, some people refer to this as 'bimeasurable'. For want of a better term, we will do so here.

Later, we'll identify appropriate notions of "morphisms" and "isomorphisms" of partitions. For now, let's just take a first look at the question and come up with a minimally-constrained working definition of a map between partitions.

One possible approach is to demand that the induced powerset map $f'|_B$ take B to B'. This proves unnecessarily restrictive. It requires that each class of B maps surjectively to an entire class of B'.

We still would allow $f'|_B$ to be non-injective, however. It can map more than one class of B to a given class of B'.

At bare minimum, we need f to provide an unambiguous way to identify some class of B' with each class of B. I.e., it must induce a map $g : B \to B'$. However, g need not be $f'|_B$ and still need not be injective. We do want g to be surjective, however. Otherwise, we're not mapping B to a partition of S'.

We could, in principle, allow a non-surjective g. However, doing so doesn't buy us anything, introduces a great deal of unnecessary ambiguity, and severely restricts our ability to reason about general properties of maps between partitions. We won't consider this case in these notes.

In order to be able to construct a g at all, no class $b \in B$ can have an image $f(b)$ that is split amongst multiple classes of B'. If it does, then we have no way of choosing which of those classes g should assign to b. However, we don't care *how* each b maps into its assigned class $g(b)$. Multiple classes of B can fill in a given $b' \in B'$ with their images, and they can do so with or without overlap. This is a choice, of course — but we will see that it is the correct one.

The case $g = f'|_B$ that we decried as overly restrictive corresponds to filling each b' with a single $f(b)$. However, it still allows the possibility (since $f'|_B$ need not be injective) of $f(b_1) = f(b_2)$ for $b_1 \neq b_2$. We're now generalizing this notion to allow multiple $f(b)$'s to fill the same b', with or without overlap. We'll later provide a plethora of definitions that distinguish various intermediate cases, along with a graph to help visualize their relationships.

We can accomplish our goal even if f is non-surjective, but such f's are problematic in certain ways that restrict the utility of allowing them. Nonetheless, we'll consider this alternative where appropriate. Unless otherwise specified, we'll assume f is surjective.

A non-surjective f only allows consideration of partitions of S' that do not have 'dead' classes. As long as every $b' \in B'$ has some $f(b)$ that sits inside it, we still obtain a surjective g. However, a surjective f allows consideration of *every* partition of S'.

Non-injectivity of f is not an issue, as long as this non-injectivity takes the right form. We'll have more to say about this shortly.

There is no need for such a caveat regarding isomorphisms, just morphisms. Any notion of isomorphism requires invertibility, and thus a bijective f. Non-surjectivity cannot come into play.

Under a given surjective map $f : S \to S'$, we'll say that (i) partition B of S **maps to** partition B' of S' if $f'(B) = B'$ and $f'|_B$ is injective and (ii) B **maps into** B' if for each $b \in B$, $f(b) \subseteq b'$ for some b' in B'. Obviously, if B maps 'to' B' it also maps 'into' B'.

Put another way, B maps 'to' B' if every class of B maps to a full, distinct class of B', and B maps 'into' B' if no class of B is split among multiple classes of B'. I.e., distinct classes of B can map to the same class of B' (fully or partly and with or without overlap), but if $f(b) \cap f(b') \neq \emptyset$ then $f(b)$ and $f(b')$ must both be subsets of the same class of B'. Because it allows overlaps, this is a generalization of the notion of coarsening. We'll visit this in more depth later.

Note that B mapping 'to' B' does *not* require f to be bijective. We're not claiming that f itself is invertible — merely that the induced map between B and B' is bijective. On a similar note, $f'|_B$ is bijective for a map 'to', but this doesn't mean we can write something like $f'^{-1}|_{B'} = (f^{-1})'|_{B'} = B$. The left side is defined and does equal B in our case — and if f is bijective, the whole expression is well-defined and correct — but f needn't be bijective. If it isn't, then $(f^{-1})'$ isn't defined.

Also note that B mapping 'into' B' is *not* the same as saying that $f'|_B$ is a (possibly noninjective) map $B \to B'$ (i.e. that $f'(B)$ has image B' or, equivalently, that $f''(B) = B'$). This would be equivalent to requiring that f map B 'to' a coarsening of B'. As mentioned, the notion of mapping 'into' is more flexible than that.

Ex. Let $S = \{1,2,3,4\}$, $S' = \{a,b,c\}$, and $B = \{(1,2),(3,4)\}$. Case 1: $B' = \{(a),(b,c)\}$, and $f:(1,2,3,4) \to (a,a,b,c)$. This is a map 'to' B', and B' looks like B and is bijective with it. Case 2: $B' = \{(a,b,c)\}$ and $f:(1,2,3,4) \to (a,a,b,c)$. This is a map 'into' where B' happens to look like a coarsening of B (i.e. $f'|_B$ is a refinement of B', even though $g \neq f'|_B$ is a noninjective map $B \to B'$). Note that the same f as in case 1 is 'to' one partition and only 'into' another. Case 3: $B' = \{(a,b,c)\}$ and $f:(1,2,3,4) \to (a,b,b,c)$. This is a map 'into' B' which is not like a coarsening of B. In fact, $f'(B) = \{(a,b),(b,c)\}$ is not even a partition. Case 4: $B' = \{(a,b),(c)\}$ and $f:(1,2,3,4) \to (a,c,b,b)$. This is not a map 'into'. $f'(B) = \{(a,c),(b)\}$, which splits the class $(1,2)$ across two classes of B'.

We'll often denote by g the induced map $b \to [f(b)]$, where $[f(b)]$ denotes the unique class of B' containing $f(b)$. We'll also employ h_b to denote the map $f|_b$ viewed relative to target set $g(b)$, and occasionally we'll use h for the collection $\{h_b; b \in B\}$. However, we won't be consistent about this — and we'll often use g and h to denote generic maps as well (ex. $g \circ f = h$). Our usage will be clear from the context.

We'll sometimes refer to the requirement that each $f(b) \subseteq b'$ for some b' as the "inclusion condition".

Under a particular surjective f, any given partition B always maps 'into' at least the trivial partition of S' — though it potentially could map 'into' various other partitions as well. Under f, B need not map 'to' any partition of S'. If it does, then that partition is unique.

The only candidate for B to map 'to' is $f'(B) = \{f(b); b \in B\}$. As our example illustrated, this may or may not be a partition of S'. Though the surjectivity of f guarantees that $f'(B)$ covers S', its elements need not be disjoint. Even if they are (and $f'(B)$ therefore is a partition), we also need that $f(b) \neq f(b')$ if $b \neq b'$. I.e., the $f(b)$'s cannot overlap partly or totally.

Intuitively, we can glom the images of classes together in various ways to form partitions of S'. However, depending on S, S', and B, we may or may not be able to find a partition B' which fully reproduces B. As we will see later, the cardinalities of B and B' and of their classes must cooperate.

Prop 1.2: Under a given surjective $f: S \to S'$: (i) f maps any given B 'to' at most one B', (ii) f maps any given B 'into' at least one B', (iii) f maps 'to' any given B' from exactly one B, (iv) f maps 'into' any given B' from at least one B, (v) f maps every B 'into' the trivial partition of S', (vi) f maps the trivial partition of S 'to' the trivial partition of S' and does not map it 'into' any other partition, (vii) f maps the singleton partition of S 'into' every partition of S', (viii) f can map 'to' the singleton partition of S' only from the singleton partition of S, and does so iff f is injective.

Note that it is quite possible for other partitions of S to map 'into' and even 'to' the singleton partition of S'. Of course, if one does map 'to' it then no other can (by (iii)) under that f. The reason that the dual doesn't hold (involving the trivial partition of S mapping 'into' other partitions, etc) is because we required f to be surjective. This breaks the duality but protects us from lots of problematic ambiguity, as we'll discuss later.

Pf: (i) Since we need $f(b) = b'$ and an injective $f'|_B$, the only candidate is $B' = \{f(b); b \in B\}$. However, if f is not injective it is quite possible that $f(b_1) \cap f(b_2) \neq \emptyset$ for $b_1 \neq b_2$.

Pf: (ii,v) The trivial partition of S' furnishes an example. $f(b) \subseteq S'$ for all $b \in B$, so B maps 'into' the trivial partition of S'.

Pf: (iii) We'll construct this in proposition 1.7 as the pull-back. Specifically, $B = \{f^{-1}(b'); b' \in B'\}$ does the trick. Because we required f to be surjective, each $f^{-1}(b')$ is nonempty. f^{-1} honors intersections and unions, so we get a partition that is patently bijective with B'. Suppose there exists some other partition B_1 that maps 'to' B' as well. Then f induces a bijective map $f'|_{B_1}$ between partitions B_1 and B'. Consider a given $b_1 \in B_1$. Since our map is 'to', $f(b_1) = b'$ for some $b' \in B'$. From set theory, we know that $b_1 \subseteq f^{-1}(f(b_1))$, so $b_1 \subseteq f^{-1}(b')$. I.e., each class of B_1 must be a subset of a class of B. This means that B_1 is a refinement (proper or not) of B. However, $f'|_{B_1}^{-1} \circ f'|_B$ establishes a bijection between B and B_1. It is perfectly possible to have a bijection between a partition and a proper refinement if both are infinite — but it is *not* possible for that bijection to effect the refinement. By definition, the coarsening map from B_1 to B must be noninjective. In our case, $f'|_B^{-1} \circ f'|_{B_1}$ is a coarsening map since each $b_1 \subset b$ for some b. This means that both $f'|_{B_1}$ and $f'|_B$ cannot be bijective to B' (or the composition $f'|_B^{-1} \circ f'|_{B_1}$ would be too, which we know it is not). So we must have $B = B_1$.

Pf: (iv,vii) The singleton partition of S maps into every partition, since $f(x)$ cannot span two classes of B'. Surjectivity of f guarantees surjectivity of g.

Pf: (vi) Since f is surjective, $f(S) = S'$ and we have a map 'to'. Suppose $\{S\}$ mapped 'into' some other partition B'. Then $f(S) \subseteq b'$ for some b'. Since B' is not the trivial partition of S', it has at least 2 classes. The other class has no inverse image, violating our assumption of surjectivity.

Pf: (viii) Let B be the singleton partition of S. Since $f(x)$ must be a single point of S', $f'|_B$ must be a set of singletons of S'. Since f is surjective, this is a partition. It therefore must be the singleton partition. However, we don't have a map 'to' (just a map 'into') unless $f'|_B$ is injective. This happens iff f itself (which is the same map, but on S rather than the set of singletons) is injective.

We also can say something useful about coarsening/refinement relations amongst partitions.

Prop 1.3: Let $f : S \to S'$ be surjective. Then (i) if f maps B 'into' B', and B_R is a refinement of B on S, then f maps B_R 'into' B', (ii) if f maps B 'into' B', and B'_C is a coarsening of B' on S', then f maps B 'into' B'_C, (iii) if f maps B 'to' B' then B' is a refinement of every partition that f maps B 'into', and (iv) if f maps B 'to' B' then B is a coarsening of every partition f maps 'into' B'.

I.e., making the source partition finer or the target partition coarser preserves the notion of being mapped 'into'.

Pf: (i) Given any $b \in B_R$, there is some $b_1 \in B$ s.t. $b \subseteq b_1$ (since it's a refinement). Since f maps B 'into' B', there is some $b' \in B'$ s.t. $f(b_1) \subseteq b'$. Since $f(b) \subseteq f(b_1)$, this means $f(b) \subseteq b'$ and f maps B_R 'into' B'.

Pf: (ii) Given any $b \in B$, there is some $b' \in B'$ s.t. $f(b) \subseteq b'$ (since f maps B 'into' B'). But $b' \subseteq b'_1$ for some $b'_1 \in B'_C$ since B'_C is a coarsening of B'. This means that $f(b) \subseteq b'_1$, so f maps B 'into' B'_C.

Pf: (iii) Suppose f maps B 'to' B', and let f map B 'into' B'_1. Consider $b' \in B'$. Since f maps B 'to' B', $f'|_B$ is injective and $f^{-1}(b') = b$ for some $b \in B$. But f maps B 'into' B'_1, so $f(b) \subseteq b'_1$ for some $b'_1 \in B'_1$. So $b' \subseteq b'_1$. Therefore, B' is a refinement of B'_1.

Pf: (iv) Suppose f maps B 'to' B', and let f map B_1 'into' B'. Consider $b_1 \in B_1$. Since f maps B_1 'into' B', $f(b_1) \subseteq b'$ for some $b' \in B'$. Consider $f^{-1}(b')$. Since f maps B 'to' B', $f^{-1}(b') = b$ for some $b \in B$. However, $b_1 \subseteq f^{-1}(f(b_1)) \subseteq f^{-1}(b')$, so $b_1 \subseteq b$, and B_1 is a refinement of B.

Combined with proposition 1.2, this is a pretty strong result and gives us the lay of the land for a specific f when it comes to maps 'into' and maps 'to'. Every B' has a unique partner which f maps 'to' it, and every partition that f maps 'into' B is a refinement of this. Suppose we have a given f. Any B which is mapped 'to' some partition B' has that B' as a unique partner, and every partition that f maps B 'into' is a coarsening of B'. In that case, we have two 'poles' and everything in between (coarser than B' or finer than B) maps into a pole or is mapped 'into' by a pole. However, the caveat 'any B' in the second half is important.

It may be tempting to think that (for the given f) this establishes a bijection between $Par(S)$ and $Par(S')$ of partitions which act as such poles. This clearly cannot be the case, since $Par(S)$ and $Par(S')$ can have different cardinalities. The problem is the 'any B'. Every B' has a unique B that maps 'to' it (which we'll see later is the pullback), but not every B has a unique B' that it maps 'to'. If such a B' exists, its pullback (i.e. partner the other way) is B — but such a B' need not always exist. If it does always exist, then f must be a bijection and indeed induces a bijection between $Par(S)$ and $Par(S')$. Let's now elaborate on this.

If, under a given surjective f, *every* partition B has a counterpart that it maps 'to', then f must be bijective. This is a very strict condition. The following proposition tells us that all of the associated maps listed above must be bijective as a consequence. It also provides a powerful converse: f is bijective iff f'' takes $Par(S)$ to $Par(S')$.

Note that the statement that the set $f'(B)$ is a partition of S' (i.e. $f''(B) \in Par(S)$) is *not* the same as the statement that f maps B 'to' its image $f'(B)$. If $f(b) = f(b')$ for distinct classes b and b', then f does not map B 'to' $f'(B)$, even though the image of the partition is a partition and the image of each class is a class. Mapping 'to' also requires that $f'|_B$ be injective.

Prop 1.4: For $|S'| > 1$ and a given $f : S \to S'$, the following three conditions are equivalent: (a) f is bijective, (b) $f''(Par(S)) \subseteq Par(S')$, and (c) $f''(CPar(S)) \subseteq CPar(S')$. If any (and thus all) hold, the following maps are well-defined and bijective (relative to their stated targets): (i) $f''|_{Par(S)} : Par(S) \to Par(S')$, (ii) $f''|_{CPar(S)} : CPar(S) \to CPar(S')$, (iii) for every partition $B \in Par(S)$, $f'|_B : B \to f'(B)$, and (iv) for every class b in every partition B in $Par(S)$, $f|_b : b \to f(b)$, and (v) we have equality in (b) and (c), meaning $f''(Par(S)) = Par(S')$, and $f''(CPar(S)) = CPar(S')$.

Note that this implies that if f is bijective, f'' maps partitions to partitions and countable partitions to countable partitions. Obviously, if a countable partition maps to a partition, the latter must be countable as well. The point is that we need only test the set of countable partitions.

The $|S'| > 1$ condition is necessary, because if $|S'| = 1$ and f is surjective then f'' automatically takes every partition of S to the trivial partition of S', even if f is noninjective.

Although conditions (b) and (c) are stated using $\subseteq$, the proposition tells us (in (v)) that proper subsets are not possible. If $f''(Par(S)) \subseteq Par(S')$ then $f''(Par(S)) = Par(S')$ and ditto for $CPar$. However, this is a *result* of the proposition, so we retain the $\subseteq$ in the statement of (b) and (c).

As mentioned, condition (b) is *not* directly equivalent to saying that every partition of S maps 'to' some partition of S' in our sense of the word 'to' (as opposed to the plain set-theory sense). Our condition for mapping 'to' is stricter than merely having $f'(B)$ be a partition. However, as a *result* of this proposition, that is indeed the case. The fact that *every* $f'(B)$ is a partition (as opposed to just, say, $f'(B)$ for some particular B we're examining) forces f to be bijective. While a bijective f can take a partition to a partition non-injectively (ex. as a coarse-graining), every B has a unique B' that it maps 'to' under such an f. Later on, we'll refer to this as its "isomorphic partner" under f.

Pf: (equivalence of (a) and (b)): (forward) Suppose f is bijective. Proposition 1.1 tells us that f' and f'' are bijective too, so $f''|_{Par(S)}$ is bijective with its image $f''(Par(S))$. We'll now show that this image lies in $Par(S')$. Consider any $B \in Par(S)$ and any $b_1, b_2 \in B$. Since f is a bijection, $f(b) \cap f(b') = \emptyset$ iff $b \cap b' = \emptyset$. Since f is surjective, we know that $f(S) = S'$, so the set $B' \equiv \{f(b); b \in B\}$ must cover S'. A disjoint cover is a partition, so $B' \in Par(S')$. This tells us that $f''(Par(S)) \subseteq Par(S')$. (backward) Suppose f is not bijective. If f is not surjective, then no $f''(B)$ can be a partition since it won't cover S'. Let us suppose that f is surjective but not injective. Then we can always find a partition B s.t. $f''(B)$ is not a partition, as we will now demonstrate. Since we required that $|S'| > 1$, we must have $|S| > 1$ as well, or f couldn't be surjective. Since f is non-injective, we must have $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$. Let $x' \equiv f(x_1) = f(x_2)$. Since $|S'| > 1$, we can pick some $y' \neq x'$. Moreover, $f^{-1}(y') \neq \emptyset$, since f is surjective. Let $b_1 \equiv \{x_1\} \cup f^{-1}(y')$ and $b_2 \equiv S - b_1$ and $B \equiv \{b_1, b_2\}$. Clearly, $x_2 \in b_2$. I.e., we picked a partition with two classes that split x_1 and x_2 and have different, overlapping images. Since y' only appears in $f(b_1)$ but x' appears in both $f(b_1)$ and $f(b_2)$, $f(b_1) \neq f(b_2)$ but $f(b_1) \cap f(b_2) \neq \emptyset$. Therefore, $f''(B)$ is not a partition, and $f''(Par(S)) \nsubseteq Par(S')$.

Pf: (equivalence of (a) and (c)): (forward) Suppose f is bijective. If B is countable, then $f'(B)$ must be countable as well. We already know from the proof of (a)=(b) that $f''(B)$ is a partition, so it is a countable partition and thus an element of $CPar(S')$. We therefore have $f''(CPar(S)) \subseteq CPar(S')$. (backward) Suppose f is not bijective. Our two-class partition B in the proof for (a)=(b) is countable. Therefore, we have shown that $f''(CPar(S)) \nsubseteq CPar(S')$.

Pf: (i-ii,v): If f is bijective, we know that $f''(Par(S)) \subseteq Par(S')$. We also know that f^{-1} is bijective, so $(f^{-1})''(Par(S')) \subseteq Par(S)$. From Proposition 1.1, we have $(f^{-1})'' = (f'')^{-1}$, so $(f'')^{-1}(Par(S')) \subseteq Par(S)$. Suppose that $f''(Par(S)) \subset Par(S')$ (proper subset). Then there is some $B' \in Par(S')$ s.t. no B maps to it under f''. However, $f''^{-1}(B')$ is a single element of 2^{2^S} since f'' is bijective, so $(f'')^{-1}(B') \in Par(S)$ since $(f'')^{-1}(Par(S')) \subseteq Par(S)$. So $f''(Par(S)) = Par(S')$. The same exact argument shows that $f''(CPar(S)) = CPar(S')$.

Pf: (iii-iv): If we have an injective map, any restriction of it is injective. Any map is surjective relative to its image, so all we really need is the injectivity of f and f'. It follows that $f'|_B : B \to f'(B)$ is bijective relative to $f'(B)$, giving us (iii), and $f|_b : b \to f(b)$ is bijective relative to $f(b)$, giving us (iv).

This is a very powerful result. All we need to exhibit is a single counterexample — some B s.t. $f''(B)$ isn't a partition — and we know that f can't be bijective. On the other hand, we need only show that all countable partitions map to partitions (which necessarily are countable) in order to guarantee that f *is* bijective. Moreover, bijectivity of f implies bijectivity of all the associated maps (relative to the appropriate targets, of course).

This also tells us that if there is some partition B s.t. $f''(B)$ is not a partition, then there also is some countable partition B_C s.t. $f''(B_C)$ is not a partition. If a counterexample exists, a countable one does too.

What if f isn't bijective? Proposition 1.4 tells us that there exists a partition B s.t. $f''(B)$ isn't a partition. However, this doesn't mean that f'' can't take some partitions to partitions.

As the following proposition tells us, our notions of ‘maps to’ and ‘maps into’ compose as expected.

Prop 1.5: Given surjective $f_1 : S_1 \to S_2$, surjective $f_2 : S_2 \to S_3$, and partitions B_1 of S_1 and B_2 of S_2 and B_3 of S_3: (i) if f_1 maps B_1 ‘into’ B_2 and f_2 maps B_2 ‘into’ B_3, then $f_2 \circ f_1$ maps B_1 ‘into’ B_3 and (ii) if f_1 maps B_1 ‘to’ B_2 and f_2 maps B_2 ‘to’ B_3, then $f_2 \circ f_1$ maps B_1 ‘to’ B_3.

I.e., the composition of maps 'into' is a map 'into' and the composition of maps 'to' is a map 'to'. The composition of a map 'into' with a map 'to' (or vice versa) is a map 'into'.

Pf: (i) Surjective maps compose to a surjective map, so we're fine on that front. Consider $b_1 \in B_1$. There exists some $b'_2 \in B_2$ s.t. $f_1(b_1) \subseteq b'_2$. There also exists some $b'_3 \in B_3$ s.t. $f_2(b'_2) \subseteq b'_3$. So $f_2(f_1(b_1)) \subseteq f_2(b'_2) \subseteq b'_3$ and $(f_2 \circ f_1)(b_1) \subseteq b'_3$. Therefore, $f_2 \circ f_1$ maps B_1 'into' B_3.

Pf: (ii) From proposition 1.1, we know that ${f_2}'|_{B_2} \circ {f_1}'|_{B_1} = (f_2 \circ f_1)'|_{B_1}$. Since the former two are bijective, their composition is too. So $(f_2 \circ f_1)'|_{B_1}$ is a bijective map $B_1 \to B_3$. Therefore, $f_2 \circ f_1$ maps B_1 'to' B_3.

We'll have a lot more to say about maps between partitions when we discuss morphisms and isomorphisms. A ‘morphism’ will correspond to our notion of a map ‘into’. However, an isomorphism won't quite correspond to our notion of a map ‘to’. It will need to be bijective as well, to ensure invertibility.

1.4.1. *Addendum: What if we don't require f to be surjective?* Let's relax our definitions of ‘map into’ and ‘map to’ to allow for a non-surjective f. We'll refer to these as a ‘flexible map into’ and a ‘flexible map to’:

- **Flexible map into**: Each $f(b) \subseteq b'$ for some b', $g = [f(b)]$ (as before), and g is surjective to B'.
- **Flexible map to**: A ‘flexible map into’ for which g is bijective.

  Simply relaxing the requirement for surjectivity of f in the definition of 'map to' accomplishes nothing. A 'map to' requires that $g = f'|_B$ and g is bijective to B', which guarantees that f is surjective (though, as discussed, it needn't be bijective). If we want to accommodate a nonsurjective f, we need to revise the definition more substantially. Our 'flexible map to' accomplishes this.

Ex. Let $S = \{1, 2, 3\}$ and $S' = \{1, 2, 3, 4\}$ and $B = \{(1, 2), (3)\}$ and $B' = \{(1, 2), (3, 4)\}$ and $B'' = \{(1, 2, 3, 4)\}$. The map $f : (1, 2, 3) \to (1, 2, 3)$ is a flexible map from B 'to' B'. It is also a flexible map from B 'into' B''.

Obviously, any ‘map into’ is a ‘flexible map into’ and any ‘map to’ is a ‘flexible map to’.

Let's see how our results so far change if we allow f to be nonsurjective, and replace ‘map to’ with ‘flexible map to’ and ‘map into’ with ‘flexible map into’.

Proposition 1.2 partly holds and partly changes when we remove the surjective f requirement and use our new definitions.

- (i) changes. A given B can ‘flexibly map to’ more than one B', one B', or none at all.

  Since f *can* be surjective (but need not), the old (i) is subsumed. There are cases, where B maps 'to' (and thus flexibly maps 'to') one partition and cases where it maps 'to' none (and since flexibly maps 'to' is the same as maps 'to' in this case, that holds for flexibly maps 'to' as well). However, if f isn't surjective it is possible for B to flexibly map 'to' more than one. For example, suppose B flexibly maps 'to' B'. Then it also flexibly maps 'to' any partition obtained from B' by moving unmapped-to elements of S' between classes of B' (but *not* forming any new classes out of them).

- (ii) still holds.

  The trivial partition of S' still works as an example.

- (iii) changes. f flexibly maps to 'B' from *at most* one B.

  The only candidate remains the pullback of B', given by $B = \{f^{-1}(b'); b' \in B'\}$. This B is a partition, but it is possible that $f'(B)$ is not. If $f^{-1}(b') = \emptyset$ for some b', then that will be the case — and f won't even flexibly map B 'into' B', let alone flexibly map 'to' it.

- (iv) changes. f need not flexibly map 'into' B' from any B.

  The reasoning from (iii) applies here too. If $f^{-1}(b') = \emptyset$ for some $b' \in B'$, then *no* partition B on S can flexibly map 'into' B', because the induced g would be non-surjective.

- (v) still holds.
- (vi) still holds.
- (vii) changes. f flexibly maps the singleton partition of S 'into' every partition of S' *for which every* $f^{-1}(b') \neq \emptyset$.

  Once again, if $f^{-1}(b') = \emptyset$, we're out of luck.

- (viii) changes in multiple ways. f can flexibly map the singleton partition 'to' non-singleton partitions. f flexibly maps the singleton partition 'to' the singleton partition iff it is bijective.

  f can (possibly) flexibly map the singleton partition of S 'to' any partition of cardinality $|B'| \leq |S|$. For example, choose any partition with $|B'| \leq |S|$ that contains at least one image point of f in each class. On the other hand, if f is not surjective, then some $f^{-1}(x') = \emptyset$, and there is no way to flexibly map 'into' the singleton partition of S', let alone flexibly map 'to' it.

  Ex. let $S = \{1, 2\}$, $S' = \{1, 2, 3, 4\}$, and $B' = \{(1, 2), (3, 4)\}$. The non-surjective map $f : (1, 2) \to (1, 3)$ maps the singleton partition of S 'to' B'.

Proposition 1.3 mostly ceases to hold if we remove the surjective f requirement and use our new definitions.

- (i) no longer holds.

  Suppose f flexibly maps B 'into' B'. If B'_R allocates unmapped-to space into a new class of its own, then f^{-1} of that class is $\emptyset$, and f cannot flexibly map B 'into' B'_R.

  Ex. let $S = \{1, 2\}$ and $S' = \{1, 2, 3, 4\}$ and $B = \{(1, 2)\}$ and $B' = \{(1, 2, 3, 4)\}$, and let $f : (1, 2) \to (1, 2)$. Then f flexibly maps B 'into' B'. Let $B'_R = \{(1, 2), (3, 4)\}$. Then $f^{-1}(3, 4) = \emptyset$, so f does not flexibly map B 'into' B'_R.

- (ii) still holds.
- (iii) no longer holds.

  Ex. let $S = \{1, 2\}$, $S' = \{1, 2, 3, 4\}$, and $B' = \{(1, 2), (3, 4)\}$. The non-surjective map $f : (1, 2) \to (1, 3)$ flexibly maps the singleton partition of S 'to' B'. However, it also flexibly maps the singleton partition of S 'to' $B'' = \{(1, 4), (2, 3)\}$. Neither B' nor B'' is a refinement of the other.

- (iv) no longer holds.

  Ex. let $S = S' = \{1, 2, 3, 4\}$ and $B' = \{(1, 2), (3, 4)\}$ and $f : (1, 2, 3, 4) \to (1, 1, 3, 3)$. Then f flexibly maps both $B_1 = \{(1), (2), (3), (4)\}$ and $B_2 = \{(1, 2), (3, 4)\}$ 'to' B', even though B_1 is not a coarsening of B_2.

Proposition 1.4 doesn't involve maps 'to' or maps 'into', so our change has no bearing on it.

Proposition 1.5 still holds. Composition of the same type is of that type.

Pf: (i) Let f_1 flexibly map B_1 'into' B_2 and let f_2 flexibly map B_2 'into' B_3. Consider $b_1 \in B_1$. We know that $f_1(b_1) \subseteq b_2$ for some $b_2 \in B_2$ and $f_2(b_2) \subseteq b_3$ for some b_3 in B_3. Therefore $f_2(f_1(b_1)) \subseteq f_2(b_2) \subseteq b_3$, so we're good on that front. Now, consider some $b_3 \in B_3$. There is some b_2 s.t. $f_2(b_2) \subseteq b_3$ and there is some b_1 s.t. $f_1(b_1) \subseteq b_2$. Therefore $f_2(f_1(b_1)) \subseteq b_3$. I.e., the induced composite g is surjective.

Pf: (ii) Upgrade f_1 and f_2 to flexibly map 'to' in our proof of (i). This means that the induced g_1 and g_2 are bijective. The induced composite g is $g_2 \circ g_1$, and therefore bijective. So we get a flexible map 'to'.

1.5. **Combinatorics.** We may wish to count the partitions, countable partitions, and finite partitions of a set.

As a matter of nomenclature, the term "partition" is used in both number theory and set theory. In the former case, it means a way of summing integers to a given value. Both involve combinatoric calculations, but they differ. We'll always mean set theoretic partitions.

For a finite set S of n elements the three are the same. The number of partitions containing m classes of sizes $n_1 \ldots n_m$ (with $\sum n_i = n$) is just the multinomial coefficient $\left(\frac{n!}{n_1! \cdot \cdots \cdot n_m!}\right)$. However, the total number of partitions of n elements — and even the number of partitions of n elements into m nonempty classes — has no closed form in terms of the usual functions.

We allow $n_i = 0$, with the usual convention that $0 \neq 1$.

The total number of ways to partition an n-object set is called the Bell number B_n, the first few of which are $B_0 = B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, and $B_5 = 52$. As is common with combinatoric functions, they increase quickly. The number of ways to partition an n-object set into m nonempty classes (i.e. precisely m classes) is $S(n, m)$, the "Stirling number of the second kind."

The Bell numbers should not be confused with the "partition numbers", which arise in number theory. The B_n notation is unfortunate in light of our use of B's for partitions. However, this won't be cause for confusion. We will not deal with Bell numbers at all and exclusively will use B's for partitions.

The following cardinality results will be of some use to us[1]. For a set S with cardinality $|S| = \beth_n$ (with $n \geq 0$):

Recall that $\aleph_0 = \beth_0$ is the cardinality of the integers, and $\beth_n \equiv 2^{\beth_{n-1}}$ for $n > 0$ (i.e. it is the cardinality of the powerset of a set with cardinality $\beth_{n-1}$). If we assume the generalized continuum hypothesis, then $\aleph_n = \beth_n$ for $n > 0$ as well. Also recall that "denumerable" or "denumerably infinite" means of cardinality $\aleph_0$, while "countable" means finite or denumerably infinite (i.e. cardinality $\leq \aleph_0$).

- For $n \geq 0$, the set of all subsets as well as the sets of finite, denumerable, countable, or all partitions, all have cardinality $\beth_{n+1}$.
- For $n > 0$, the sets of finite, countable, and denumerable subsets all have cardinality $\beth_n$.
- For $n = 0$ (i.e. $\beth_0 = \aleph_0$), the set of finite subsets has cardinality $\aleph_0$ and all other of the above sets (including the set of finite partitions) have cardinality $\beth_1$.

  This means that the sets of countable or denumerable subsets are $\beth_1$ for *both* $|S| = \aleph_0$ and $|S| = \beth_1$.
- $m^{\beth_n} = \beth_{n+1}$ for any $1 < m \leq \beth_{n+1}$. More generally, for infinite m, $\beth_j^{\beth_i} = \beth_{\max(i+1,j)}$.
- $m \cdot \beth_n = \max(m, \beth_n)$ (whether m is finite or infinite).
- $\beth_n + \beth_m = \beth_{\max(n,m)}$.

1.6. **Full Intersections.** Suppose we start with a set S, an index set I, and a (nonempty) set $Z = \{X_i; i \in I\}$ of sets X_i of subsets of S s.t. each X_i covers S.

I.e., each X_i is a set of subsets of S that cover S. For example, we could pick a partition as one of the X_i's.

We can think of Z as a map $I \to 2^{2^S}$.

[1]See my notes on cardinality at `https://kmhalpern.com/2011/08/02/cardinality/` for a detailed discussion of cardinalities and proofs of all of these cases.

Though I need not be countable, we'll write X_1, X_2, etc for convenience of presentation.

To avoid an excess of caveats, we'll always assume that any set of sets (i.e. X_i) and any set of sets of sets (i.e. Z) is nonempty and consists solely of nonempty sets. If we're allowing empty sets in any capacity, we'll say so explicitly.

We define the set of **full intersections** of Z to be all nonempty intersections that select a single element from each X_i. We'll denote this set $F(Z)$.

As a point of terminology, we'll use the term "full intersection" to refer to two distinct things: (i) the specific choice of an element of each X_i, and (ii) the set which results from their intersection. Formally, we can think of (i) as an element of $\prod_{i\in I} X_i$. Each of these is distinct, whereas their intersections need not be. If we write $\cap_Z : \prod_{i\in I} X_i \to 2^S$ as the map that takes each of (i) to the corresponding (ii) by performing the intersection $\cap x_i$, it is quite possible that $\cap_Z$ is not injective (and it certainly need not be surjective). The choice of meaning (i) or (ii) will be clear from the context or will be explicitly stated. For convenience, we'll sometimes refer to the elements of $\prod_{i\in I} X_i$ (i.e. (i)) as "full intersection specs" (or "specs" for short) and the sets they intersect to (i.e. (ii)) as "full intersection sets" (or "intersections" for short) where the distinction needs to be made. $F(Z)$ refers to the set of nonempty full intersection sets, not the set $\prod X_i$. Formally, $F(Z) = \{\cap_{i\in I}\pi_i(x); x \in \prod_{i\in I} X_i\}$ minus any empty intersection if present (and where $\pi_i(x)$ denotes the projection of the i^{th} component of x).

Ex. suppose we have $X_1 = \{a, b, c\}$ and $X_2 = \{a, d, e\}$ and $X_3 = \{b, c, f, g\}$, where $a \ldots g$ are subsets of S. There are 36 full intersection specs for Z (i.e. $X_1 \times X_2 \times X_3$ has 36 elements). These include things like (a, d, f) and (c, e, b). Obviously, some of the resulting intersections are the same. Ex. (b, a, c) and (c, a, b) are distinct specs with the same intersection. If X_1 and X_2 and X_3 are partitions, then $b\cup c = d\cup e = S-a$ and $f \cup g = a$. It is easy to see that the only specs which can have nonempty intersections are (a, a, f), (a, a, g), (b, d, b), (b, e, b), (c, d, c), and (c, e, c). Of these, only the first two and at least two of the latter four are guaranteed not to be empty. I.e., the 36 distinct specs produce only between 4 and 6 distinct nonempty intersections.

For certain purposes (such as working with σ-algebras, which are closed under countable intersections and complements, but not arbitrary intersections), it is convenient to define an analogous notion constrained to countable intersections. We'll term this the set of **countable full intersections** of Z and denote it $F_C(Z)$. However, we must be careful how we define it.

Suppose we define it as all intersections involving at most $\aleph_0$ choices of sets (each from a distinct X_i). Then, we're including all finite intersections. For $|Z| = \aleph_0$ (and thus $|I| = \aleph_0$), $F_C(Z)$ would be the set of all intersections, not just full intersections. Even confining ourselves to denumerably infinite sets of intersections doesn't help. For one thing, it excludes the finite $|Z|$ case altogether. But there also are improper denumerably infinite subsets of I which don't equal I. Ex. if $I = N$, we could have a denumerably infinite intersection which only involves elements of the X_{2n}.

We'll define $F_C(Z)$ as follows: (i) for $|Z| \le \aleph_0$, $F_C(Z) \equiv F(Z)$, and (ii) for $|Z| > \aleph_0$, $F_C(Z)$ is the set of all nonempty intersection sets obtained from intersection specs involving a denumerably infinite number of distinct X_i's. For convenience, we'll refer to the relevant specs and intersections as "countable" even though they aren't allowed to be finite in the case of $|Z| > \aleph_0$.

I.e., to obtain a spec in (ii), we begin by selecting a denumerably infinite subset $Y \subset Z$. Then we select a single x_i from each $X_i \in Y$. $F_C(Z)$ contains all nonempty intersection sets obtained from such specs (varying the X_i's within a Y and the Y's within Z). Ex., if $|Z| = \aleph_1$, and we index its elements by $\mathbb{R}$, then each choice of Y corresponds to a monotonically increasing function $f : N \to \mathbb{R}$ (where N is the natural numbers). Each spec corresponds to a choice of such f along with a denumerably infinite sequence $\{x_i\}$ s.t. $x_i \in X_{f(i)}$. Any $\cap_i x_i$ that is nonempty is in $F_C(Z)$.

The distinction between $F(Z)$ and this $F_C(Z)$ is similar to that between direct products and direct sums of vector spaces. Those are the same for finite products, but for an infinite product the direct sum consists of all tuples with only a finite number of nonzero entries while the direct product consists of all tuples. In our case, countable takes the place of finite and we can think of S as taking the place of zero.

Though it is important to be consistent, we shouldn't fret too much over the particulars of the definition. Conceptually, the definition of $F_C(Z)$ we chose is useful when studying those σ-algebras which can be generated from partitions via unions (i.e. the σ-algebras with "partition bases"). In that case, the X's will be partitions and we'll be seeking their common refinement. For finite and denumerably infinite Z's, we'll want to take the full intersections because those form the common refinement. Allowing partial intersections (drawing a single element from each of a subset of the X_i's) won't buy us anything because those partial intersections can be written as countable unions of full intersections. For uncountable Z's we'd ideally prefer to take the full intersections, but the σ-algebra is only closed under countable intersections. In this case, we're limited, and this creates all sorts of problems. Ultimately, it doesn't matter precisely which definition of $F_C(Z)$ we pick. We introduce it primarily to illustrate why certain things *don't* work, and all of its possible definitions exhibit suitable failings. Otherwise, we'd just pick the one which didn't and avoid these problems altogether. Unfortunately, there is no such workable choice. As a result (and as we'll see when we discuss σ-algebras in a future piece), the utility of partition bases for σ-algebras is severely limited.

Note that, in general, $F_C(Z)$ is neither a subset nor a superset of $F(Z)$. It is a distinct set. Only for $|Z| \leq \aleph_0$ is it (by definition) equal.

Bear in mind that part (i) of the definition only applies if Z is countable, not S. It is perfectly possible to have an uncountable Z and a countable S, in which case (ii) would apply. It is also possible (and more common) to have a countable Z and an uncountable S, in which case (i) would apply.

In general, not all full intersection specs of Z yield nonempty or distinct intersections. Some can be empty and some can be redundant. However, both $F(Z)$ and $F_C(Z)$ always cover S.

In our earlier example, the 36 distinct specs could yield anywhere from 0 to 36 distinct, nonempty sets. When we specified that the X's were partitions, we narrowed this to between 4 and 6 distinct nonempty sets.

Why must they cover $F(Z)$ and $F_C(Z)$? Since we required that each X_i covers S (and Z is nonempty by assumption), $F(Z)$ and $F_C(Z)$ must be nonempty. Given any $x \in S$, we can find some set x_i in each X_i that contains x. If X_i isn't a partition, there may be more that one such x_i, so we'll just pick one. The intersection of all these chosen x_i's (one from each X_i) is in $F(Z)$ and contains x. If $|Z| > \aleph_0$, pick any denumerably infinite subset of these x_i's. Their intersection is in $F_C(Z)$ and contains x. Therefore, x is contained in some element of $F(Z)$ and some element of $F_C(Z)$.

Note that neither F nor F_C is a function from $2^{2^{2^S}}$ to 2^{2^S}. Each takes a set of sets of sets (an element of $2^{2^{2^S}}$) and produces a set of sets (an element of 2^{2^S}). However, the sets of sets they can take as input are constrained to cover S and be nonempty (as is the set of sets of sets). I.e., F and F_C are defined only on a subset of $2^{2^{2^S}}$. This makes them "partial functions", rather than functions, on $2^{2^{2^S}}$.

As the following proposition tells us, if all the X_i's are partitions of S, then the full intersection specs cannot yield duplicate intersection sets (but their intersection sets still can be empty).

Prop 1.6: If all the X_i's are partitions of S, then (i) $F(Z)$ is a partition of S, (ii) $F(Z)$ is a refinement of every $X_i \in Z$, and (iii) there are no redundant full intersection specs, in the sense that no two full intersection specs produce the same *nonempty* full intersection set.

Pf: (i) Consider any point $x \in S$. Each $X_i \equiv \{s_i^1, s_i^2, \dots\}$ is a partition, so x is in one and only one $s_i^{j_i} \in X_i$ for each i. Let $s \equiv \cap_i s_i^{j_i}$ using this choice of $s_i^{j_i}$ for each i. Since x is in every $s_i^{j_i}$, it is in their intersection. By construction, this is the only full intersection spec whose resulting set contains x. Any other has at least one $s_i^{j'_i}$ which does not contain x. Therefore, every x is in the full intersection set resulting from one and only one spec. I.e., the nonempty full intersection sets form a partition of S. (ii) Consider some X_i. Any $s_i^{j_i} \in X_i$ can be written as $s_i^{j_i} = \cup_{j_1,\dots j_{i-1},j_{i+1},\dots}(s_1^{j_1} \cap \dots \cap s_{i-1}^{j_{j-1}} \cap S_i^{j_i} \cap s_{i+1}^{j_{j+1}} \dots)$. I.e., we fix $s_i^{j_i}$ but range over all elements of every other X_j. [Many of these intersections are empty, but we don't care; they simply do not contribute to the union.] Put another way, we take the union of all full intersection sets that resulted from specs involving our particular $s_i^{j_i}$ (as opposed to another element of X_i). Every other X_j is a partition, so this union clearly is just $s_i^{j_i}$. I.e., every element of every X_i can be written as a union of full intersections. $F(Z)$ therefore constitutes a refinement of X_i. (iii) We saw in (i) that every point $x \in S$ sits in a set defined by a unique full intersection spec, which we explicitly constructed. Therefore, no two specs can yield the same (nonempty) set.

This is a general result and holds regardless of the cardinalities involved. However, we must be careful how we interpret it. Proposition 1.6 is exclusively a statement about nonempty sets. If S is a measure space, there is no analogous result regarding sets of nonzero measure. It is possible for every element of every X_i to have nonzero measure, but for some of the full intersections to have zero measure. We are taking a limit, and this can result in a set of measure zero. Even in the finite case, we must be wary of measure. The intersection of two sets with nonzero measure can have zero measure. Ex. consider $([0,1] \cup X) \cap ([2,3] \cup X)$, where X is the rational numbers contained in $(1,2)$.

Note that an analogous result does *not* hold for $F_C(Z)$. Like $F(Z)$, $F_C(Z)$ must cover S (whether or not Z consists of partitions). However, when $|Z| > \aleph_0$ the elements of $F_C(Z)$ need not be disjoint, even if Z consists of partitions. We'll see an example of this shortly.

We'll typically use B_i for the elements of Z when they are partitions, and X_i when they are not (or if it is not certain).

We'll say that a set of full intersection specs is **realized** if no two full intersection specs yield the same set and no spec yields the empty set. If Z consists of partitions, proposition 1.6 tells us that two specs can't yield the same nonempty set, so "realized" in that case just means that each spec yields a nonempty set.

For brevity, we'll sometimes say that Z or $F(Z)$ is "realized". We always mean that the relevant set of full intersection specs is realized.

Obviously, if Z is realized, every (nonempty) subset of Z is realized as well.

If Z consists of partitions, then proposition 1.6 tells us that $F(Z)$ is a partition. If Z does not consist of partitions, then $F(Z)$ need not be a partition. However, in *both* cases, Z cannot be realized unless $\prod |X_i| = |F(Z)|$, because no two specs can produce the same nonempty intersection. If $F(Z)$ is finite, then this is an iff — but if infinite sets are involved, it is possible to have $\prod |X_i| = |F(Z)|$ for an unrealized Z.

There is a related construction which we already encountered in an example and will see a lot more of. Given any (nonempty) set X of subsets of S, we can construct a corresponding set of partitions $B(X) \equiv \{\{s, \bar{s}\}; s \in S\}$ (where it is understood that if any $s = \Omega$, its empty complement is omitted from the corresponding partition). I.e., for each set in X, we create a binary partition consisting of it and its complement. We then can define $F(B(X))$ and $F_C(B(X))$ just as for any other set of partitions. We'll abbreviate these $F(X)$ and $F_C(X)$.

The context will make the meaning clear. Suppose we encounter something like $F(foo)$. If foo is a set of sets of sets (ex. a set of partitions), then we mean the usual. However, if foo is just a set of sets, we mean $F(B(foo))$.

Obviously, if X is a partition, then $F(X) = X$. Unless X is the trivial partition, $F(X)$ is not realized. [If X is the trivial partition $\{S\}$, then $F(X)$ is realized but vapid.] Note that $F_C(X)$ for a partition need *not* equal X. For example, if X consists of the singleton partition of $\mathbb{R}$, then each complement has a single hole in it. Any denumerable intersection of these yields a countable set of missing elements, so we still get an uncountable set. There is no requirement that a denumerable intersection must include one of the singleton sets and not just consist of complements.

In order for $F(X)$ in this construction to be realized, both the sets in X and their complements need to large. This is because *every* combination of them has to overlap to avoid nullification. We'll see a similar phenomenon when we discuss the "internal copy" of a direct product of partitions. Our $B(X)$ construction can be thought of as a special case of that, where the partitions are all binary. As a visual example, consider $\mathbb{R}^3$. If we slice it into half-spaces along the three axes (i.e. using the $x-y$ plane, $y-z$-plane, and $x-z$ plane), we get 3 partitions for which all 8 full intersection specs yield nonempty sets.

1.7. **Order on Set of Partitions.**

There is a natural partial order on the set $Par(S)$ of all partitions of S. It orders partitions by refinement.

We cannot inherit the subset order from 2^{2^S} because neither a subset nor a superset of a partition is a partition. The former would not cover S and the latter would not be disjoint.

The convention is to define $B \leq B'$ if B is a (possibly improper) refinement of B'.

This is somewhat counterintuitive, because $|B| \geq |B'|$ when $B \leq B'$ by this definition. When we discuss the lattice of partitions, the resulting join and meet operations will nonetheless feel like they should (i.e. join will feel like a product and meet will feel like a sum). However, because of this convention, we'll need to invert the lattice of partitions when comparing it with certain other lattices (ex. the lattice of σ-algebras). There is no problem doing so, but this is something to be cognizant of when discussing equivalences of lattices, etc.

The reason for the counterintuitive choice of direction comes from equivalence relations. A relation aRb on S can be written as a subset $R \subseteq S \times S$. As we mentioned earlier, partitions and equivalence relations form a one-to-one correspondence. An equivalence relation is a type of relation, and thus is a subset $R \subseteq S \times S$. If B is a refinement of B' then $\sim$ is stricter that $\sim'$, and fewer ordered pairs satisfy it. I.e. $R \subseteq R'$. The partition order corresponds to the subset order on $S \times S$ of the associated equivalence relations, and this is where the choice of direction comes from.

Under this partial order, the minimum partition is the most refined and the maximum is the least refined. I.e., the minimum is the singleton partition and the maximum is the trivial partition.

As we will see, this partial order, along with certain natural join and meet operations, turn $Par(S)$ into a complete lattice.

1.8. **Pullback of a Partition.** We can pull back a partition but not push one forward.

Prop 1.7: Given sets S and S', map $f : S \to S'$, and partition B' of S', we have a partition f^*B' of S obtained via $f^*B' \equiv \{f^{-1}(b'); b' \in B'\}$.

Pf: $f^{-1}(b'_1 \cap b'_2) = f^{-1}(b'_1) \cap f^{-1}(b'_2)$, so disjointness is preserved. Since $f^{-1}(S') = S$ and $f^{-1}(S') = f^{-1}(\cup_{b' \in B} b') = \cup_{b' \in B} f^{-1}(b')$, the set f^*B', thus defined, covers S. It therefore is a disjoint cover of S and hence a partition of S.

The push-forward of a partition B of S fails to cover S' if f is not surjective and may fail to be disjoint if f is not injective. If f is surjective but not injective, the push-forward of B can be converted into a partition of S' by merging overlapping images of classes of B. This will be the $G(f(B))$ construct we describe later.

Prop 1.8: Suppose $f : S \to S'$ is surjective. Then (i) f maps B 'to' B' iff $f^*B' = B$. (iii) f maps B 'into' B' iff f^*B' is a coarsening (proper or not) of B.

I.e., for a given B', f^*B' is the only partition of S that f maps 'to' B', and is a coarsening of every partition that f maps 'into' B'.

Pf: (i-backward) Consider class $b \in f^*B'$. Since $b = f^{-1}(b')$ for some b', and since $f(f^{-1}(s')) \subseteq s'$ for sets in general, $f(f^{-1}(b')) \subseteq b'$ and we satisfy the condition for mapping 'into'. In fact, since f is surjective, $f(f^{-1}(b')) = b'$. It is easy to see that $f'|_B$ is injective, making the map 'to'. Suppose $b_1 \neq b_2$, and let $b_1 = f^{-1}(b'_1)$ and $b_2 = f^{-1}(b'_2)$. If $b'_1 = b'_2$ then $b_1 = b_2$, violating our premise — so $b'_1 \neq b'_2$. However, B' is a partition, so this means $b'_1 \cap b'_2 = \emptyset$. We just saw that $f(f^{-1}(b')) = b'$ since f is surjective. Therefore, $f(b_1) = b'_1$ and $f(b_2) = b'_2$, and $f(b_1) \neq f(b_2)$. We therefore have a map 'to'.

Pf: (i-forward) Let f map B to B'. Then $f'(B) = B'$ and $f'|_B$ is injective. Consider $f^{-1}(b')$ for some $b' \in B'$. Since $f'(B) = B'$, $f^{-1}(b')$ must be a union of classes of B, and since $f'|_B$ is injective, we know that this union consists of a single class. Therefore, $f^{-1}(b') \in B$, so $f^*B' \subseteq B$. Suppose $b \in B$. Since $f'(B) = B'$, $f(b) \in B'$. But $f'|_B$ is injective, so $f^{-1}(f(b))$ can only equal b. Therefore, $b \in f^*B'$, and $B \subseteq f^*B'$.

Pf: (ii) Let f map B into B'. Then $f(b) \subseteq b'$ for some $b' \in B'$ for each $b \in B$. Consider $f^{-1}(b')$ and suppose that it isn't a union of classes of B. Since B is a partition, this means there is some b s.t. only part of b appears in $f^{-1}(b')$. However, $f(b)$ sits in a single class of B', so that class must be b'. Therefore, $f^{-1}(b')$ must include all of b. Since it cannot contain a piece of a class of B, $f^{-1}(b')$ must be a union of classes of B. I.e., f^*B' is a coarsening (proper or not) of B. Now, let's go the other way. Suppose that B is a refinement of f^*B'. For any $b \in B$, $b \subseteq b_1$ for some $b_1 \in f^*B'$, so $f(b) \subseteq f(b_1) = f(f^{-1}(b'))$ for some $b' \in B'$. However, $f(f^{-1}(b')) \subseteq b'$ (with equality if f is surjective, which it is).

If we allow f to be non-surjective, things change substantially. Proposition 1.8 no longer applies. We still pull back a partition B' to a partition $B \equiv f^*B'$, but it is possible that f does not flexibly map 'to' or 'into' B'.

Ex. let $S = S' = \{1,2,3,4\}$, let $B' = \{(1),(2),(3),(4)\}$, and let $f : (1,2,3,4) \to (1,2,3,3)$. Then $f^*B' = \{(1),(2),(3,4)\}$. However, f maps this to $\{(1),(2),(3)\}$, which is not a partition of S'.

Given any function $f : S \to S'$, let $B'_{S'}$ be the singlet partition on S'. A special case of the pullback is $f^*B'_{S'}$. This is the set of inverse images of points in S', and is the finest partition that can be a pullback (along f) of any partition of S'.

1.9. **Product of Partitions.** Suppose we wish to define a "product-like'' operation amongst partitions. There are several approaches we can take.

We'll soon define an addition-like operation as well, but it is more complicated to describe.

Let B_I be a set of partitions indexed by some linearly-ordered index set I, and let S_I denote the corresponding sets which are partitioned by them (i.e. B_1 is a partition of S_1, B_2 is a partition of S_2, etc). Several candidates immediately present themselves:

As before, we'll write things in terms of $B_1, B_2, \ldots$ for convenience even though I need not be countable.

- Cartesian Product: This takes an arbitrary B_I and corresponding S_I and returns a partition B of $S \equiv S_1 \times S_2 \times \ldots$ (i.e. the usual direct product of sets $S \equiv \bigtimes_{i \in I} S_i$). The elements of S are tuples with the slots labeled by I and with an element of S_i in the i^{th} slot. We define B via the usual cartesian set product (induced on 2^S): $B \equiv B_1 \times B_2 \times \ldots$.

  Ex. If $S_1 = S_2 = \mathbb{R}$ and both B_1 and B_2 are unit-interval partitions, then $B_1 \times B_2$ is a partition of $\mathbb{R}^2$ into unit squares.

  Ex. If $|I| = n$ and $|B_i| = m$ for all i, then $|B| = m^n$.

- Pullback: This takes an arbitrary S_I and a partition B_i on one specific S_i and returns an "internal copy" of B_i as a partition of $S \equiv \bigtimes_{i \in I} S_i$. We define the resulting B'_i to consist of S_j in all slots except the i^{th}. I.e. $B'_i \equiv (\bigtimes_{j<i} S_j) \times B_i \times (\bigtimes_{j>i} S_j)$.

  I.e., we can think of it as a cartesian product where, other than B_i, each B_j is the trivial partition of its respective S_j.

  The name is not inconsistent with out earlier usage of the term 'pullback'. The B'_i thus defined is nothing other than the pullback of B_i along π_i, where $\pi_i : S \to S_i$ projects out S_i from S. Proposition 1.7 tells us that this $B'_i = \pi_i^* B_i$ is indeed a partition of S.

  Ex. In our previous $\mathbb{R}$ example, B'_1 consists of unit strips perpendicular to the y-axis and B'_2 consists of unit strips perpendicular to the x-axis. I.e. $B'_1 = \{([0,1),\mathbb{R}), ([1,2),\mathbb{R}), \ldots\}$ and $B'_2 = \{\mathbb{R}, ([0,1)), (\mathbb{R}, [1,2)), \ldots\}$.

  Ex. If $|I| = n$ and $|B_i| = m$, then $|B'_i| = m$.

  This is convenient because it takes partitions of distinct sets S_i and converts them to partitions of a common S, thus allowing apples-to-apples comparisons. Suppose we start with a set of partitions B_I on corresponding sets S_I. We can form the cartesian product to get a single partition B or we can take all the pullbacks to get a set of B'_i's. If $|I| = n$ and each B_i has m elements, then the cartesian product has m^n elements and the set of pull-backs has $m \cdot n$ elements. Note that our original information content consists of $m \cdot n$ elements, so we lose nothing. The cartesian product just introduces a lot of duplicative bloat.

- Direct product: This takes a set of partitions B_I all on the same S (i.e. all the S_i's are the same) and produces their common refinement B on that same S. Specifically, $B = F(B_I)$. I.e., it is the set of all nonempty full intersection sets. We'll denote this $B * B'$ or $B_1 * \ldots$, etc.

  This should not be confused with the direct product of sets, which yields the Cartesian product.

- Direct sum: This is analogous to the direct product, but produces $F_C(B_I)$ instead of $F(B_I)$. I.e., it is the set of all nonempty countable full intersections (as we chose to define F_C earlier). We'll denote it $B \oplus B'$ or $B_1 \oplus \ldots$, etc.

> We call this the "direct sum" because it has a similar flavor, but this is nonstandard terminology and notation. For example, in linear algebra, the direct sum has zeroes in all but a finite (rather than countable) number of slots.

We'll predominantly work with the direct product and pullback of partitions. We'll refer to the direct product $B = B_1 * B_2 \cdots$ as a **product of partitions**, and we'll refer to the pullback of B_i as its **internal version** and the indexed set of all pullbacks $B'_I \equiv \{B'_i; i \in I\}$ as the **internal version** of B_I.

> Note that the "internal version" of B_i is not intrinsic to B_i. It depends on the entirety of S_I (but not the rest of B_I). The internal version is always relative to a specific S_I, even if the notation B'_i omits this.

Prop 1.9: Given B_I and S_I, let $S = \bigtimes_{i \in I} S_i$. Then the direct product of the pullbacks equals the cartesian product of B_I. Formally, $\bigtimes_{i \in I} B_i = F(B'_I)$.

> In words, the product of pullback partitions is the same as the cartesian product of the original partitions.

> Note that we cannot speak of $F(B_I)$ because the B_i's are partitions of different sets. The pullbacks create a view of them on the same set S and we then can take the product of partitions.

> Ex. Consider our earlier $S_1 = S_2 = \mathbb{R}$ example, with $B_1 = B_2$ the unit intervals. We saw that $B_1 \times B_2$ is a partition of $\mathbb{R}^2$ into unit squares, and B'_1 and B'_2 are partitions of $\mathbb{R}^2$ into unit strips parallel to the axes. $B'_1 * B'_2$ is their common refinement and is a partition of $\mathbb{R}^2$ into unit squares. This is useful because, unlike B_1 and B_2, which exist on distinct copies of $\mathbb{R}$ (which happen to be identical in this example, but need not be in general), B'_1, B'_2, and $B_1 \times B_2 = B'_1 * B'_2$ all are partitions of the same $S = \mathbb{R}^2$, and we therefore can compare them.

> One takeaway from our example is that the classes of the internal view (i.e. pullback) tend to be very large. This is no surprise given its definition. We include all of S_j in every slot except the i^{th}. The pullbacks are partitions in only one of the directions and include the full set in every other.

> One reason we prefer a product of pullbacks to a cartesian product is that $B * B'$ is a binary operation on $Par(S)$, whereas the cartesian product is not. As we will see, that binary operation turns out to be our lattice join.

> Pf: The proof is straightforward conceptually, and we just need to keep track of the pieces. First, consider a specific element $b \in \bigtimes_{i \in I} B_i$. This is of the form $(b_1, b_2, \ldots,)$ with $b_i \in B_i$. Next, consider a specific element $b' \in F(B'_I)$. Each $B'_j = \{S_1 \times \cdots \times S_{j-1} \times b_j \times S_{j+1} \times \ldots; b_j \in B_j\}$. An element $b' \in F(B'_I)$ is a full intersection of $\{B'_1, B'_2, \ldots\}$, whose spec contains one class from each B'_j. I.e., we're choosing a specific $b_j \in B_j$ for each j, resulting in $b' = (b_1, b_2, \ldots,)$ with $b_j \in B_j$. It is patently clear that the sets $\bigtimes_{i \in I} B_i$ and $F(B'_I)$ are the same. Each choice of $(b_1, b_2, \ldots)$ is in both $F(B'_I)$ and $\bigtimes_{i \in I} B_i$, and no other set is in either.

Note that it is quite possible to have $B_1 * B_2 = B_3 * B_4$. A given B can be a product of partitions in multiple different ways.

> Ex. If $S = \{1,2,3,4,5,6\}$ and $B_1 = \{(1,2),(3,4),(5,6)\}$ and $B_2 = \{(1,6),(2,3),(4,5)\}$ and $B_3 = \{(2,4),(3,6),(1,5)\}$, then $B_1 * B_2 = B_2 * B_3 = B_1 * B_3 = 2^S$.

We'll say that a partition B is **irreducible** if there exist no nontrivial partitions B_1 and B_2 s.t. $B = B_1 * B_2$, and we'll say it is **realizably irreducible** if there exist no nontrivial partitions B_1 and B_2 s.t. $B = B_1 * B_2$ and $\{B_1, B_2\}$ is realized. Obviously, irreducible implies realizably irreducible. As we'll soon see, the two concepts are in fact the same.

The grammar of this can be misleading, and we must be very careful about negations. There are three disjoint possibilities for B. Let U denote the statement "There exist one or more choices of nontrivial B_1 and B_2 s.t. $B = B_1 * B_2$, but none are realized", let V denote the statement "There exist one or more choices of nontrivial B_1 and B_2 s.t. $B = B_1 * B_2$, and at least one of these is realized", and let W denote the statement "There exist no choices of nontrivial B_1 and B_2 s.t. $B = B_1 * B_2$." Clearly, one and only one of these can be true. 'Irreducible' is W and 'realizably irreducible is $W \vee U$. It would make sense to refer to V as 'realizably reducible'. 'Reducible' or 'not irreducible' corresponds to $\neg W$, which (since $U \vee V \vee W$ always is true), is the same as $(U \vee V)$. 'Not realizably irreducible' is $\neg(W \vee U) = \neg W \wedge \neg U$. Since $U \vee V \vee W$ always is true, this is the same as V. I.e., 'not realizably irreducible' is indeed the same as 'realizably reducible'. Since irreducible implies realizably irreducible, the contrapositive holds too. We've shown that it is perfectly fine to phrase this as 'realizably reducible' implies 'reducible' — which makes sense. I.e., even though 'realizably irreducible' is a single noun rather than an adjective-noun pair, the logical relationships involved allow us to pretend that 'realizably' is an adjective — even when applied to irreducible, which ordinarily would make no sense.

Prop 1.10: (i) A finite B is realizably irreducible iff $|B|$ is prime. (ii) Given a non-prime finite B with cardinality $|B| = \prod p_i^{i_j}$ (with the p_i's prime), it has realized decompositions into realizably irreducible (i.e. prime cardinality) components $B_1, \ldots, B_m$, and all such decompositions have the same cardinality $m = \sum_i i_j$.

I.e., any finite non-prime B can be written as a 'maximal' realized product of irreducible components AND all such maximal realized products have the same number of components.

Pf: (i) For a prime B to be realizably irreducible, we need $B = B_1 * B_2$, where $F(\{B_1, B_2\})$ is realized. This means $|B| = |B_1| \cdot |B_2|$ (which would not be the case if $F(\{B_1, B_2\})$ were not realized). Since $|B|$ is prime and we've excluded the trivial partition (the only partition of size 1), this is impossible. (ii) Consider a non-prime finite B with $|B| = \prod_j p_j^{i_j}$. Let $m \equiv \sum_j i_j$. Arrange B in an m-dimensional array, with sides corresponding to the p's (i.e. i_1 of the sides are of size p_1, i_2 of the sides are of size p_2, etc). List the sides in some order, and let m_l ($l = 1 \ldots m$) be the length of side l according to this order (i.e. the associated p_j for that side, with each p_j appearing i_j times amongst the m_l's in this capacity). Denote by $b_{k_1,\ldots,k_m}$ the elements of B arranged this way (i.e. k_l runs from 1 to m_l). Define $b_k^l \equiv \cup_{k_1,\ldots,k_{l-1},k_{l+1},\ldots,k_m} b_{k_1,\ldots,k_{l-1},k,k_{l+1},\ldots,k_m}$, where each k_i runs from 1 to m_i. I.e., we take the union over all b's with k in the l^{th} slot. Define $B_l \equiv \{b_k^l; k = 1 \ldots m_l\}$. By construction, this is a partition because each $b_{k_1 \cdots k_m}$ appears in one and only one b_k^l and each $x \in S$ appears in one and only one $b_{k_1 \ldots k_m}$. Visually, each B_l slices up the m-dimensional array into $(m-1)$-dimensional arrays perpendicular to the l^{th} axis. The product of all the B_l's is, by construction, B (each full intersection selects one slot from each B_l, thus homing in on a single $b_{k_1 \ldots k_m}$). Also by construction, no full intersection is empty. So $B = B_1 * \cdots * B_m$ and $F(\{B_1, \ldots, B_m\})$ is realized. Moreover, any such construction necessarily must have cardinality m because we must land at the same prime factorization each way. Otherwise, there remain reducible components.

Prop 1.11: If B is (countably or uncountably) infinite, then, given any choice of cardinality $\gamma \leq |B|$, we can factor B into some B_1 and B_2 (i.e. $B = B_1 * B_2$) s.t. $|B_1| = |B|$ and $|B_2| = \gamma$ and $F(\{B_1, B_2\})$ is realized.

Pf: The proof follows the same basic idea as in the finite case. Let I be an index set of cardinality $|B|$, let $\beta : I \to B$ be a labeling bijection, and let I' be an index set of cardinality $\gamma \leq |B|$. For any infinite I and $0 < |I'| \leq |I|$, $|I \times I'| = |I|$. The two therefore are bijective, and we can pick a bijection $\alpha : I \times I' \to I$ (the actual choice is irrelevant to us). Define $b_{ij} \equiv \beta(\alpha(i,j))$, where $i \in I$ and $j \in I'$. Visually, we can think of an $|I| \times |I'|$ array (where the sides can be uncountably infinite). We're just rearranging the elements of B into $|I|$ rows of length γ. We then define $b_i^1 \equiv \cup_j b_{ij}$ and $b_j^2 \equiv \cup_i b_{ij}$ and $B_1 \equiv \{b_i^1; i \in I\}$ and $B_2 \equiv \{b_j^2; j \in I'\}$. As before, these are partitions and $B_1 * B_2$ is the set of their full intersections. It equals B for the same reason as before (we're just selecting i and j by picking an element of B_1 and an element of B_2), and $F(\{B_1, B_2\})$ is realized for the same reason too.

As the proof shows, we not only can factor B for any such choice of γ, but we can do so in infinitely many ways.

These two propositions tell us a few things:

- Irreducible and realizably irreducible are equivalent concepts, and therefore reducible and realizably reducible also are equivalent concepts (see our earlier discussion of the grammar and logic of this).

> By definition, irreducible implies realizably irreducible. Proposition 1.11 tells us that an infinite B is neither, so they are vacuously equivalent in that case. Let B be finite. Proposition 1.10 tells us that any B with prime $|B|$ automatically is irreducible and therefore also realizably irreducible. Again they are equivalent in this case. If B has nonprime $|B|$, then this same proposition tells us that it is realizably reducible and therefore reducible. In all three cases, B is either both realizably reducible and reducible or both realizably irreducible and irreducible. The two therefore are equivalent.

- When B is infinite, there is no meaningful notion of irreducibility. Any infinite B can be reduced ad infinitum.
- When B is finite, there can be many distinct factorizations of a given reducible B into irreducible partitions.
- When B is finite, all factorizations of a given reducible B into irreducible partitions have the same cardinality.
- Very few partitions are irreducible. We need $|B|$ to be finite and prime.

1.10. **Cardinalities of Intersection Sets.** The cardinalities of $F(Z)$ and $F_C(Z)$ can be a bit unintuitive, so let's briefly consider them. Obviously, these will depend heavily on the specifics of Z and the underlying set S. However, we can establish a few rules, particularly in the case of partitions.

If S is finite, then Z must be too, and so must $F_C(Z) = F(Z)$. There is nothing unexpected there.

Suppose S is infinite, with $|S| = \beth_n$ $(n \geq 0)$. If Z is finite, then we again have $F_C(Z) = F(Z)$. It may be tempting to think that both are finite in that case, but they need not be. Each X_i can be infinite.

> Ex. Let S be the integers, let B_1 be the singleton partition, let B_2 be the partition $\{(-\infty, 0), [0, \infty)\}$, and let $Z = \{B_1, B_2\}$. Then $F(Z) = B_1$ is infinite.

For a general Z, the cardinality of each $|X_i| \leq \beth_{n+1}$ since there are at most $\beth_{n+1}$ subsets of S. There are at most 2^{2^S} possible X_i's, so $|Z| \leq 2^{2^{\beth_n}} = \beth_{n+2}$. In the maximal case, there are $\beth_{n+1}^{\beth_{n+2}} = \beth_{n+3}$ specs. However, the cardinality of $F(Z)$ cannot exceed $\beth_{n+1}$, since it too is a set of sets and thus has $|F(Z)| \leq |2^S|$. In the maximal case — or in any case with large X_i's and/or a large Z — the vast majority of specs produce either duplicate intersections or the empty set.

> When defining $F(Z)$, we required the elements of X_i to be covers of S. Technically, the maximum cardinality of Z therefore will be that of the set of all covers of S (let's call it $Cov(S)$) rather than that of the set of all sets of subsets (i.e. 2^{2^S}) — many of which are not covers of S. However, it turns out that $|Cov(S)| = |2^{2^S}| = \beta_{n+2}$, rendering the distinction irrelevant for our purposes. The cardinality of $Cov(S)$ is clearly bounded above by that of 2^{2^S}, so $|Cov(S)| \leq \beth_{n+2}$. Since every partition is a cover, $Par(S) \subseteq Cov(S)$. This alone doesn't get us what we need, though, because $|Par(S)| = \beth_{n+1}$. Consider some partition B of S. It contains at most $\beth_n$ sets, since we can't have more classes than elements of S. Define $Y \equiv 2^S - B$, the set of all sets that aren't classes of B. Since $|2^S| = \beth_{n+1}$ and $|B| \leq \beth_n$, $|Y| = \beth_{n+1}$. The set of sets of subsets of S that don't contain any classes of B is just 2^Y. This has cardinality $\beth_{n+2}$, just like 2^{2^S}. Now, construct a set of covers $\{B \cup u; u \in (2^Y - \emptyset)\}$. I.e., we tack a nonempty element of Y onto B to get a (non-partition) cover. Since Y contains none of the elements of B, any two such covers are distinct elements of 2^{2^S}. I.e., we have at least $|Y|$ distinct covers, so $|Y| \leq |Cov(S)|$, and $|Cov(S)|$ is bounded below by $\beth_{n+2}$ as well.

For a Z in which all the X_i's are partitions, things are a little different. Each $|X_i| \leq \beth_n$, since the singleton partition is a refinement of all other partitions. There are $\beth_{n+1}$ partitions of S, so $|Z| \leq \beth_{n+1}$. In the maximal case, there are $\beth_n^{\beth_{n+1}} = \beth_{n+2}$ specs. However, by proposition 1.6 $F(Z)$ is a partition, so its cardinality cannot exceed that of S. I.e., $F(Z) \leq \beth_n$. Once again, in the maximal case — or in any case with large X_i's and/or a large Z — the vast majority of specs produce either duplicate intersections or the empty set.

We can use similar reasoning for $F_C(Z)$. In the general case, there are $\beth_{n+2}$ denumerable specs. $F_C(Z)$ still is constrained to be a set of subsets, and therefore has $|F_C(Z)| \leq \beth_{n+1}$. When the X_i's are partitions, there are $\beth_{n+1}$ denumerable specs. However, $F_C(Z)$ need not be a partition, so we can only constrain it to a set of subsets. I.e., $|F_C(Z)| \leq \beth_{n+1}$.

We compute in the general case as follows: let each $|X_i| = \beth_{n+1}$ and let $|Z| = \beth_{n+2}$. There are $\beth_{n+2}$ denumerable subsets of Z (i.e. choices of $\aleph_0$ X_i's). Each of these yields $\beth_{n+1}^{\aleph_0} = \beth_{n+1}$ specs, so we have $\beth_{n+1} \cdot \beth_{n+2} = \beth_{n+2}$ total denumerable specs. For the case of partitions, let each $|X_i| = \beth_n$ and let $|Z| = \beth_{n+1}$. Once again, there are $\beth_{n+1}$ denumerable subsets of Z, and each of these produces $\beth_n^{\aleph_0}$ specs. For $n > 0$, this gives us $\beth_n$ and we get $\beth_n \cdot \beth_{n+1} = \beth_{n+1}$ total specs. For $n = 0$, we get $\aleph_0^{\aleph_0} = \beth_1$ specs per selection, and $\beth_1 \cdot \beth_1 = \beth_1$ total specs. In both cases, the result is $\beth_{n+1}$ total specs.

Bear in mind that $F(Z)$ and $F_C(Z)$ can be completely distinct sets, regardless of their relative cardinalities. It is quite possible for $F_C(Z)$ to be smaller than, equal to, or greater than $F(Z)$ in size. It is also possible for each of $F(Z)$ and $F_C(Z)$ to (simultaneously) have elements the other does not. However, although $F(Z)$ can be a proper subset of $F_C(Z)$, it is not possible for $F_C(Z)$ to be a proper subset of $F(Z)$.

It is easy to see why $F_C(Z)$ can't be a proper subset of $F(Z)$. $F_C(Z)$ must cover S. Consider any $x \in S$. Any intersection spec (countable or not) must have at least one nonempty intersection containing x. We simply pick whichever class of each partition contains x. Since $F(Z)$ must be a partition of S, no proper subset of it can cover S.

Note that we need to be a little careful when constructing examples where $F(Z)$ isn't a subset of $F_C(Z)$. Let $s \in F(Z)$. Some set of full intersection specs produce it. However, if s itself appears in any of those specs, then it also appears in $F_C(Z)$. We can construct from any of these full specs a countable spec which results in s, simply by including s and any other countable subset of the full spec. To find examples where $s \in F(Z)$ but isn't in $F_C(Z)$, we therefore need to look at 'limits' — infinite intersections that don't include s itself as one of the sets in the spec.

Case 1 ($F(Z) = F_C(Z)$, trivially): They are equal (by definition) when $|Z| \leq \aleph_0$.

Case 2 ($F(Z)$ proper subset of $F_C(Z)$ and smaller cardinality): Let $S = N$ (the nonnegative integers), and let Z consist of all 2-class partitions of S. I.e. $Z = \{(s, N-s); \emptyset \subset s \subset N\}$. Although S is countable, Z is not, so part (ii) applies in the definition of $F_C(Z)$, and we can potentially have $F(Z) \neq F_C(Z)$. Since $F(Z)$ is a partition, it cannot have more elements than S. It is trivial to see that it is, in fact, just the singleton version of S. Therefore, $|F(Z)| = \aleph_0$. On the other hand, $F_C(Z)$ consists of all subsets of N that have an infinite number of missing elements. This includes all finite sets, not just the singletons, so $F(Z) \subset F_C(Z)$ is a proper subset. However, $F_C(Z)$ also includes some infinite subsets. The even integers form a subset of N that has an infinite number of missing elements, and the same is true of any nonempty subset of it. There are $|2^{|N|/2}| = \beth_1$ of these, so $|F_C(Z)| \geq \beth_1$. It can't be bigger than $\beth_1$, since it is a set of subsets of N. Therefore, $|F_C(Z)| = \beth_1$, which is larger than $|F(Z)| = \beth_0$.

Case 3 ($F(Z)$ and $F_C(Z)$ overlap but neither is a subset of the other): Let S be some set with $|S| = \beth_n$ for $n > 1$. Let $\{X, S-X\}$ be some partition of S with both $|X| = \beth_n$ and $|S-X| = \beth_n$. Define Z to consist of all (two-class) partitions of the form $\{(X-\{x\}, (S-X)\cup\{x\}); x \in X\}$. I.e., each two-class partition involves shifting a single element from X to $S-X$. Clearly, $|Z| = \beth_n$. We'll refer to the elements of each partition as $\{b_1, b_2\}$ with $b_1 = X - \{x\}$ for some $x \in X$ and $b_2 = (S-X)\cup\{x\}$. Consider any intersection spec, full or countable. If more than two b_2's appear in it, then all the b_2's reduce to $S-X$ since their extra elements differ and are washed away in the intersection. Suppose some $b_1 = X - \{x\}$ and some $b_2' = (S-X)\cup\{y\}$ appear in the same spec. $(S-X)\cap(X-\{x\}) = \emptyset$, so $b_1 \cap b_2' = (X-\{x\})\cap\{y\}$. We can't have $y = x$ since we pick only one class from each partition in the spec, so b_1 and b_2' can't be from the same partition. We therefore have $y \in X - \{x\}$, so $b_1 \cap b_2' = \{y\}$. This gives us four spec scenarios: (a) we only have b_2's, in which case we get $S-X$ as our intersection, (b) we have more than one b_2 and at least one b_1, in which case we get $\emptyset$, (c) we have all b_1's, in which case we get $X - \cup_i\{x_i\}$, where each x_i is the missing point from an included b_1 (and the index set may be uncountable, even though we're writing the index as i), and (d) we have all b_2's except one b_1, in which case we get $(X - \cup_i\{x_i\})\cup\{y\}$, where each x_i is the missing point from an included b_1 and y is the extra point from our b_2. Note that y can never equal any of the x_i's since that would involve having two classes from the same partition. So far, we've been agnostic to the size of our spec. (i) Let's compute $F(Z)$. Cases (a) and (b) always give us $S-X$ and $\emptyset$. Case (c) gives us $\emptyset$ because $\cap_{x\in X}(X-\{x\}) = X - \cup_{x\in X}\{x\} = \emptyset$. By the same token, case (d) gives us $(X - \cup_{x\neq y}\{x\})\cup\{y\} = \{y\}\cup\{y\} = \{y\}$. I.e., $F(Z)$ consists of $S-X$ along with a singleton for each element of X. (ii) Now, let's compute $F_C(Z)$. Cases (a) and (b) still give us $S-X$ and $\emptyset$. Now, however, we can only accumulate a countable number of holes in cases (c) and (d). In each case, the purely b_1 part of the intersection gives us $\cap_i(X-\{x_i\}) = X - \cup_i\{x_i\}$, where x_i denotes the missing point in the corresponding b_1. In case (d), the relevant y can't appear in the x_i's for the same reason as before. We thus get $(X-\cup_i\{x_i\})\cup\{y\} = (X - \cup_i\{x_i\})$. I.e., case (d) is subsumed by case (c). We therefore see that $F_C(Z)$ consists of $S-X$, along with every set of the form X minus a denumerably infinite number of points. Since X is uncountable, removing a countable number of points doesn't change its cardinality, and $F_C(Z)$ does not include any singletons. Specifically, $F(Z)\cap F_C(Z) = \{S-X\}$ (the one set X), $F(Z) - F_C(Z) = \{\{x\}; x \in X\}$ (i.e. all singleton sets from X), and $F_C(Z) - F(Z) = \{(X-Y); Y \subset X, |Y| = \aleph_0\}$ (i.e. all sets that are X minus $\aleph_0$ points).

If Z is realized, then $|F(Z)| = \prod |X_i|$. We therefore may deduce certain constraints on realizability from the cardinalities of the sets involved.

Note that the following are necessary but not sufficient conditions. They are requirements which must be met for $F(Z)$ to be realized, but merely meeting them does not mean that $F(Z)$ *is* realized. This would depend on the specific X_i's involved.

- Case 1: Finite S, and the X_i's are partitions:
  - ◇ If the X_i's are finite and Z is finite, then we require $\prod |X_i| \leq |S|$.

    We may worry that some classes of distinct X_i's could be the same or one could be a coarse-graining of another. However, it is easy to see that any such scenario would lead to empty intersections. We'd just pair one subset of one class with the complement of its parent. Realizability requires that no class of any X_i be a subset of any class of another X_i. We therefore need $|F(Z)| = \prod |X_i|$. Since $F(Z)$ is a partition by proposition 1.6, $|F(Z)| \leq |S|$.
  - ◇ We obviously can't have an infinite Z or infinite X_i's.
- Case 2: Finite S, and the X_i's need not be partitions:
  - ◇ If the X_i's are finite and Z is finite, then we require $\prod |X_i| \leq |2^S|$.

    $F(Z)$ need not be a partition, so we're only constrained by the number of subsets of S.
  - ◇ We obviously can't have an infinite Z or infinite X_i's.
- Case 3: $S = \beth_n$, and the X_i's are partitions:
  - ◇ If the X_i's are finite and Z is finite, cardinality imposes no constraint.

    $\prod |X_i| < |S|$ always in that case.

- ◇ If the X_i's are infinite and Z is finite, cardinality imposes no constraint.

  As a partition, each $|X_i| \le |S| = \beth_n$. However, $\beth_n^m = \beth_n$ for any finite m.

- ◇ If $Z = \beth_m$, then regardless of the size of the X_i's, cardinality demands that $|Z| \le \beth_{n-1}$.

  The X_i's are partitions, so we know that $|X_i| \le |S| = \beth_n$. In the largest case, $|X_i|^{|Z|} = \beth_n^{\beth_m} = \max(\beth_n, \beth_{m+1})$. Since $F(Z)$ must be a partition, it has cardinality $\le |S| = \beth_n$ too. This can only happen if $m \le n-1$. Note that the maximum $|F(Z)| = \beth_n$ can be achieved either by having $|Z| = \beth_{n-1}$ or by having at least one $|X_i| = \beth_n$. I.e., we can have a small product of large partitions or a large product of small partitions.

- Case 4: $S = \beth_n$ and the X_i's need not be partitions:
  - ◇ If the X_i's are finite and Z is finite, cardinality imposes no constraint.

    We require that $\prod |X_i| \le |2^S|$, which is always the case.

  - ◇ If the X_i's are infinite and Z is finite, cardinality imposes no constraint.

    As a general cover, each $|X_i| \le \beth_{n+1}$. However, $F(Z)$ now is only constrained to be a cover, so $|F(Z)| \le \beth_{n+1}$. $\beth_{n+1}^m = \beth_{n+1}$ for any finite m.

  - ◇ If $Z = \beth_m$, then regardless of the size of the X_i's, cardinality demands that $|Z| \le \beth_n$.

    The X_i's are covers, so we know that $|X_i| \le \beth_{n+1}$. In the largest case, $|X_i|^{|Z|} = \beth_n^{\beth_m} = \max(\beth_{n+1}, \beth_{m+1})$. Since $F(Z)$ is a cover, it must have cardinality $\le \beth_{n+1}$ too. This can only happen if $m \le n$. Note that the maximum $|F(Z)| = \beth_{n+1}$ can be achieved either by having $|Z| = \beth_n$ or by having at least one $|X_i| = \beth_{n+1}$. I.e., we can have a small product of large X_i's or a large product of small X_i's.

Note that it is very hard to obtain a realized $|F(Z)| = \aleph_0$. We need Z to be finite, none of the X_i's to be uncountable, and at least one of them to be $\aleph_0$. I.e., we need Z to consist of a nonzero finite number of countable X_i's and a (possibly zero) finite number of finite X_i's. Any other combination won't work.

Suppose Z is realized. A finite number of finite X_i's would yield a finite $|F(Z)|$. An infinite Z would yield at least the next higher cardinality for $F(Z)$. If any X_i is uncountable, so if $F(Z)$. There's really just one way to get a countable $F(Z)$ that is realized. Of course, if we drop the requirement that it be realized, then all bets are off.

A similar (but stricter) result holds for σ-algebras. No σ-algebra can have cardinality $\aleph_0$.

Next, let's consider the size of $F(X)$ from our binary-partition construction, and the associated realizability constraints. Given a set X of subsets of S, we defined $B(X)$ to be a set of partitions, one for each $s \in X$ and consisting solely of s and its complement.

- In the case of finite S, let $n \equiv |S|$ and let $m \equiv |X|$. Then $F(X)$ can only be realized if $2^m \le n$. I.e., we are bounded by $|X| \le \log_2 |S|$.

  Since X is an arbitrary set of subsets, we have $m \le 2^n$. The number of specs is 2^m, and this has a maximum possible value of 2^{2^n}. Since $F(X)$ is a partition, $|F(X)| \le n$. So, we need $2^m \le n$.

  This is a severe constraint. In general, the vast majority of specs yield duplicate or empty intersections.

  Note that $F(X)$ is a partition, even though X need not be. In fact, as mentioned earlier, X cannot be a (nontrivial) partition and have $F(X)$ realized. Our notation obscures that we're really working with $F(B(X))$, where $B(X)$ is a set of partitions.

  If $|S|$ isn't an even power of 2, then we must truncate. Ex. if $|S| = 5$, then $F(X)$ can be realized for $|X| = 2$ but not $|X| = 3$.

- In the case of $|S| = \beth_n$ (for $n \ge 0$), we have two cases:
  - ◇ If X is finite, cardinality imposes no constraint on realizability.

    Again, this does not guarantee that it is realized for any given X. All we're saying is that cardinality doesn't get in the way.

◇ If $|X| = \beth_m$, then $F(X)$ can only be realized if $m \leq n-1$.

Since X is a set of subsets, $m \leq n+1$. There are $2^{|X|} = \beth_{m+1}$ specs. In the maximal case of $m = n+1$, this amounts to $\beth_{n+2}$ specs. However, $F(X)$ is a partition of S, so $|F(X)| \leq \beth_n$. As before, the vast majority of specs yield duplicate or empty intersections.

Let's take a look at the two lowest cases. If $|S| = \beth_1$, we need $|X| \leq \aleph_0$ (i.e. countable) in order for $F(X)$ to have a shot at realizability. If $|S| = \aleph_0$, we need $|X|$ to be finite in order for $F(X)$ to have a shot at realizability. In that case, $|F(X)| = 2^{|X|}$ is finite as well.

Following up on our earlier comment, it is impossible to have $|F(X)| = \aleph_0$.

As we saw, the only way to get $|F(Z)| = \aleph_0$ is with a finite Z, no uncountable X_i's, and at least one $|X_i| = \aleph_0$. With $F(X)$, every partition is binary, so $|X_i| = 2$ for all $X_i \in B(X)$. Therefore, we can never have a $F(X)$ (i.e. $F(B(X))$) with cardinality $\aleph_0$.

1.11. **Lattice of Partitions.** Given a set S, the set $Par(S)$ of partitions of S forms a complete (and thus bounded) lattice.

As we saw, there is a natural partial order on $Par(S)$ by refinement, with the convention that $B \leq B'$ when B is a refinement (proper or not) of B'.

The join operation is just our product of partitions $B * B'$. However, the meet operation is a bit trickier to define. Intuitively, we want to merge any partitions of B and B' which overlap. One way to visualize this is that we place the two partitions above one another and then stitch together any class of B which overlaps with a class of B'. When all the stitching is done, the resulting connected components are the classes of our new partition.

Formally, we define $b \in B$ and $b' \in B'$ to be 'connected' if $b \cap b' \neq \emptyset$. We then define an equivalence relation on $B \cup B'$ by $a_1 \sim a_2$ (for $a_1, a_2 \in B \cup B'$) iff there exists a path between them involving connected pairs. This partitions $B_1 \cup B_2$ into sets of sets. For each class of this partition of $B_1 \cup B_2$, we take the union of all the sets in it, yielding a subset of S. These subsets of S are the classes of our new partition of S.

Another way to view the meet and join operations is in terms of equivalence classes. We mentioned earlier that we can think of a partition as an equivalence relation, aka a suitable subset $R \subset S \times S$. Let R and R' be the equivalence relations corresponding to B and B' (i.e. $(x, y) \in R$ iff x and y are in the same $b \in B$, etc). The equivalence relation lattice is ordered by subset inclusion (of the R's), its join is $R \cap R'$, and its meet is based on $R \cup R'$. Why "based on" rather than equal to $R \cup R'$? It is easy to see that $R \cap R'$ is an equivalence relation, however transitivity is an obstruction for $R \cup R'$. If $(a, b) \in R$ and $(b, c) \in R'$ but neither appears in the other, then we won't have transitivity. We can't just arbitrarily add in pairs to create transitivity, because there are many possible ways to do so. However, there is a natural minimal equivalence relation R'' which contains $R \cup R'$. We define our meet to be the relation R'' s.t. $(a, b) \in R''$ iff there is a transitivity chain from a to b in $R \cup R'$. I.e., there is some sequence of x's s.t. $(a, x_1) \in R \cup R'$, $(x_1, x_2) \in R \cup R'$, $\dots$, $(x_n, b) \in R \cup R'$. Obviously, $R \cup R' \subseteq R''$ and the reflexivity and symmetry requirements on R and R' ensure that we have symmetry and reflexivity on R'' too. Unsurprisingly, the difficulty of defining this meet operation is comparable to that of defining the meet for partitions.

There is no standard notation for the meet operation, so we'll write $B|B'$.

Prop 1.12: $B * B'$ is the coarsest partition that is a refinement of both B and B'.

Pf: We already know that $B * B'$ is a common refinement from our earlier discussion. Suppose we have any common refinement B_r. Consider a class $b_r \in B_r$. Since B_r is a common refinement of B and B', we must have $b_r \subseteq b$ and $b_r \subseteq b'$ for some $b \in B$ and $b' \in B'$. I.e., b_r is in some intersection of a class of B and a class of B'. However, $B * B'$ is precisely the set of these intersections. Therefore, b_r is in some class of $B * B'$. Any common refinement of B and B' must therefore either equal $B * B'$ or be a refinement of it.

Prop 1.13: The $*$ operation is commutative, associative, idempotent (i.e., $B * B = B$), and has the trivial partition as identity.

Pf: Commutativity and associativity follow from those of $\cap$. Idempotence is obvious, since $b_i \cap b_j = b_i$ if $i = j$ and is $\emptyset$ otherwise. $\Omega \cap b_i = b_i$, so $\{\Omega\} * B = B$.

More generally, $B * B' = B'$ iff B' is a (possibly improper) refinement of B.

Prop 1.14: $B|B'$ is the finest partition that is a coarsening of both B and B'.

Pf: From the definition of the stitching process, $B|B'$ must be a coarsening of both. Otherwise, there would be some overlap of its classes with one or the other, and further stitching would be possible. Suppose there exists some partition B'' that is a coarsening of both B and B'. Then, each class $b'' \in B''$ is both a union of classes $b'' = \cup b_i$ of B and a union of classes $b'' = \cup b'_j$ of B'. We're dealing with partitions, so these are disjoint unions. Since B'' is a partition, there can be no overlap of b'' with any other class of B''. We already knew that elements of B can't overlap with one another and the elements of B' can't overlap with one another. However, this tells us that the b_i's in the union can't overlap with any b' outside the union and the b'_j's in the union can't overlap with any b outside the union. I.e., no stitching is possible between the b_i's and b'_j's and anything else. Therefore, any set resulting from such stitching must be a subset of b''. I.e., the classes of $B|B'$ are subsets of those of B'', and $B|B'$ either equals B'' or is a refinement of it.

Prop 1.15: The | operation is commutative, associative, idempotent, and has the singleton partition as identity.

Pf: (sketch) Commutativity is evident from the symmetry of the definition. For idempotence, we note that $B|B$ involves stitching of $B \cup B$. Since B is a partition, there are no overlaps except of each b with its copy in the second B. Clearly, this means $B|B = B$. As for the singleton partition, denote it B_S. Each $b \in B$ overlaps precisely the singletons of its elements. However, the b's are disjoint from one another, and each element appears in one and only one $b \in B$. I.e., a given b connects to all the relevant singletons but no other elements of B. Taking the union of b with the singletons comprising it adds nothing. Therefore $B|B_S = B$. Associativity is easiest to visualize in terms of the corresponding equivalences. $R|R'$ is the smallest equivalence relation containing R and R'. Clearly, both $R|(R'|R'')$ and $(R|R')|R''$ are the smallest equivalence relation containing R, R', and R''. More formally, we saw that $B|B'$ is the finest partition that is a coarsening of both B and B'. Therefore, $(B|B')|B''$ is the finest partition that is a coarsening of all three. However, $B|(B'|B'')$ also has this property, so they must be the same.

More generally, $B|B' = B'$ iff B' is a (possibly improper) coarsening of B.

Neither operation has an inverse in either the algebraic or set-map sense.

Algebraically, we can't define an inverse. Since Ω is the identity for $*$, each B would need a (unique) B' s.t. $B * B' = \Omega$. However, this is impossible since $B * B'$ will be finer that both, not coarser. Similarly, $B|B'$ will be coarser than both, so we can't have $B|B' = B_S$. From the standpoint of mere set maps, the operations are $* : Par(S) \times Par(S) \rightarrow Par(S)$ and $| : Par(S) \times Par(S) \rightarrow Par(S)$. For a fixed B (i.e. locking one parameter), each acts like a map $Par(S) \rightarrow Par(S)$. However, they are not inverses of one another. $B * (B'|B) \neq B'$ and $B|(B' * B) \neq B'$. Instead, we have the similar-looking but quite distinct 'absorption' identities stated below.

Prop 1.16: $B * (B'|B) = B$ and $B|(B * B') = B$.

Pf: (i) Suppose we stitch together B' and B into $B|B'$. Let $b \in B$ and $a \in B|B'$. By construction, a either contains all of b or none of it. If it contains all of it, then $*$ gives us back b. Otherwise, we get the empty set. This makes sense, since we first took the common coarsening — which means that both B and B' are refinements of it — and then refined that back to B. (ii) A class in $B|(B * B')$ stitches classes in B with those of the form $b \cap b'$ by zig-zagging. However, only the classes of $B * B'$ within a given $b \in B$ can be stitched this way because $B * B'$ is a refinement of B. Given $b \in B$, each class in $B * B'$ either is a subset of it or disjoint with it. The former are the only ones stitched together with it, but they have the same elements as b so taking their union with it adds nothing. We therefore just get b back when we take |. This too is no surprise. We started by finding the common refinement of B and B', so the common coarsening of that with B is just B itself.

The absence of an algebraic inverse for | (which clearly would take on the role of addition) is not the only obstruction to $*$ and | forming a ring on $Par(S)$. As seen in proposition 1.16, $*$ and | are also non-distributive.

This means that the lattice is not a distributive lattice.

Ex. consider $S = \{a, b, c\}$ and let $B_1 = \{(a, b), (c)\}$, $B_2 = \{(a), (b, c)\}$ and $B_3 = \{(a, c), (b)\}$. Then $B_1 * B_2 = B_2 * B_3 = B_1 * B_3 = B_S$, with B_S the singleton partition. $B_1|B_2 = B_2|B_3 = B_1|B_3 = \{S\}$, the trivial partition. $B_1 * (B_2|B_3) = B_1 * \{S\} = B_1$, but $(B_1 * B_2)|(B_1 * B_3) = B_S|B_S = B_S$. Nor does distributivity work if we try to treat $*$ as our addition and | as our multiplication instead. In that case, $B_1|(B_2 * B_3) = B_1|B_S = B_1$, but $(B_1|B_2) * (B_1|B_3) = \{S\} * \{S\} = \{S\}$.

Commutativity, associativity, and idempotence are general properties of lattice join and meet operations, so our proofs are really just special cases. The properties in proposition 1.16 are called "absorption identities". They also are general properties of a lattice.

We could have just skipped the proofs, citing them as general lattice properties instead. However, that wouldn't yield much insight into what is going on here — so we slogged through the specifics.

Any (nonempty) set with two binary operations that are commutative, associative, idempotent, and satisfy the absorption identities is a lattice. This is an alternate way of defining a lattice (as opposed to the usual poset, meet, and join definition), one which allows us to focus on pure algebra rather than limits and bounds.

The partition lattice is bounded below by the singleton partition, which is a refinement of every partition, and bounded above by the trivial partition, which is a coarsening of every partition. This makes it a bounded lattice.

We won't delve too deeply into lattices, but we do note the following:

- The lattice of partitions is a complete lattice, which is stricter than merely being bounded.

  Recall that a complete lattice is a lattice in which every subset of the lattice (in our case, a set of partitions $\{B_i\}$) has a meet and a join which is in the lattice. It can be shown that every complete lattice is bounded, so completeness is a stricter condition. We won't prove it here, but our pairwise definitions of the meet and join operations extend to arbitrary meets and joins in the obvious way. [It is easy to see that there are no aspects of our definitions that depend on an intersection or union being only finite or countable.]

- The lattice of partitions is equivalent to the lattice of equivalence relations.

  A morphism of lattices is a map between their elements that preserves meets and joins. An equivalence (more commonly called an isomorphism) of lattices is a bijective lattice morphism whose inverse is a lattice morphism. In our case, we just map each partition to its corresponding equivalence relation. We won't prove it here, but the meets map to meets and the joins map to joins. Isomorphism means that both lattices (or neither) are complete, bounded, etc. In our case, they are both complete (and thus bounded).

1.12. **Counterpart of meet for set of sets.** We discussed a way to construct a partition from a set X of subsets of S. For each set $s \in X$, we constructed a binary partition $\{s, S - s\}$ and then took the product of all these partitions. This is the set of full intersections $F(X)$, which really means $F(B(X))$. It is very fine, in the sense a given $s \in X$ (or its complement) will likely span many classes of $F(X)$. This construction corresponds to a repeated use of our lattice join operation.

We could attempt something similar with the meet operation, but it is pretty obvious that the binary-partition approach won't be as useful. For one thing, if any two sets partly overlap, we'll just end up with the trivial partition — which then trivializes the meet with every other binary partition.

Ex. suppose $S = \{1, 2, 3, 4\}$ and $X = \{(1, 2, 3), (2, 3, 4)\}$. Then if we stitch together the partitions $\{(1, 2, 3), (4)\}$ and $\{(1), (2, 3, 4)\}$ we get $\{S\}$.

If the sets either don't overlap or only fully overlap, we're just coarsening them in the obvious way. This isn't a particularly interesting scenario. Either way, we end up with a fairly useless construction.

However, we still can define a useful version of this operation. Denote by $G(X)$ the set of sets we obtain by performing a meet-style stitching on the sets of X.

I.e., we define $s_1, s_2 \in X$ to be "connected" iff $s_1 \cap s_2 \neq \emptyset$ and $s_1 \sim s_2$ iff there exists a path from s_1 to s_2 via connected pairs. We then take the union of all the s's in each class to obtain a set in $G(X)$.

We can, in fact, obtain the lattice meet operation from this stitching via $B_1|B_2 = G(B_1 \cup B_2)$. More generally, $B_1|\ldots|B_n = G(\cup B_i)$.

I.e., we just toss all the classes of all the B's in a big pot and then perform our stitching.

Prop 1.17: Given a cover X of S, $B = G(X)$ is the finest partition of S s.t. for each $U_i \in X$, $U_i \subseteq b_j$ for some $b_j \in B$.

Pf: By construction, each $U_i \subseteq b_j$ for some $b_j \in G(X)$ and the elements of $G(X)$ must be disjoint sets of S. Since X itself is a cover, every element $s \in S$ appears in some set of $G(X)$. Therefore, $G(X)$ forms a disjoint cover of S and is a partition. Suppose that some partition B' exists such that each $U_i \subseteq b'_k$ for some $b'_k \in B'$. Consider a given $U_i \in X$. We already know that $U_i \subseteq b_j$ for some $b_j \in B_j$. We wish to show that $b_j \subseteq b'_k$. Suppose that there exists $s \in b_j$ s.t. $s \notin b'_k$. Since $U_i \subseteq b'_k$, $s \notin U_i$. Since b_j arose from our stitching procedure, s is in some $U' \in X$ s.t. $U' \sim U_i$. Moreover, since this $U' \sim U_1$, any other U' containing s will be stitched to U_1 through s, so if $s \in U$ then any U' containing s must have $U' \sim U_i$, and thus $U' \subseteq b_j$. However, from the premise of the proposition, this $U' \subseteq b'_l$ for some $b'_l \in B'$. Since B' is a partition, either $b'_l = b'_k$ or they are disjoint. If $b'_l = b'_k$, then $s \in b'_k$, violating our premise that $s \notin b'_k$. However, $s \in b'_l \cap b_k$, so they can't be disjoint. Therefore, it is impossible for s to be in b_j but not in b'_k. This means that $b_j \subseteq b'_k$. Thus B is a (possibly improper) refinement of B'. Since this is true of every B' for which the proposition's premise holds, B is the common refinement of all such partitions, and thus is the finest partition for which that premise holds.

Obviously, if X consists only of disjoint sets (and, in particular, if it is a partition of S), then $G(X) = X$.

Prop 1.18: Given a surjective $f : S \to S'$ and a partition B of S (and recalling that $f'(B) = \{f(b); b \in B\}$), (i) $G(f'(B))$ is the finest partition of S' that f maps B into and (ii) every other B' that f maps B 'into' is a coarsening of $G(f'(B))$.

Pf: (i) In order to map B 'into', each $f(b) \subseteq B'$. Proposition 1.17 tells us that $G(f'(B))$ is precisely the finest partition s.t. this condition holds. (ii) If there existed some partition B' which wasn't a coarsening of $G(f'(B))$ and for which this condition held, then $B' * G(f'(B))$ would be finer than both and still obey the condition — violating (i).

Note that even if $f'(B)$ is a partion of S' (and thus $G(f'(B)) = f'(B)$), it does *not* follow that f maps B 'to' it. This requires that g be bijective, which it may not. We could have $f(b_1) = f(b_2)$ for some $b_1 \neq b_2$.

1.13. **Topologies.** Any partition B of S is the basis for a topology T_B consisting of all arbitrary unions of its classes. This is called a **partition topology**.

Prop 1.19: Any two partition topologies on S are distinct.

Pf: Suppose B and B' are distinct partitions that are bases for the same T on S, and suppose $b \in B$ but $b \notin B'$. Then $b \cap b' \neq \emptyset$ for some $b' \in B'$ since B' covers S. Since $b \neq b'$, $b \cap b'$ is a proper subset of at least one of b and b'. As a topology, T is closed under finite intersections, so $b \cap b' \in T$ and must be expressible as a union of elements of B and a union of elements of B' (since we assumed that both are bases for T). If $b \cap b'$ is a proper subset of b' it cannot be expressed a union of element of B', and if it is a proper subset of b it cannot be expressed as a union of elements of B. I.e., it is a member of the topology that cannot be expressed as a union of sets from one of the bases (and possibly either). This violates our assumption.

Prop 1.20: $T_{B'}$ is a subspace of T_B iff B is a refinement of B'.

Pf: If B is a refinement of B' then every $b' \in B'$ is a union of elements of B, so every union of elements of b' is a union of elements of b. Therefore $T_{B'} \subseteq T_B$. Now, let's go the other way. Suppose B is not a refinement of B'. Then some $b' \in B'$ cannot be expressed as a union of elements of B. Any open set in $T_{B'}$ that has b' in its unique expression as a union of elements of B' cannot be expressed as a union of elements of B. Therefore, $T_{B'} \not\subseteq T_B$.

Not every topology is a partition topology. It is possible that none of the bases of T is a partition.

We can specify a topology in terms of its open sets or its closed sets. In terms of closed sets, arbitrary intersections are allowed. Denoting by $\bar{T}$ the closed sets of T (i.e. $\bar{T} \equiv \{\bar{o}; o \in T\}$), it may be tempting to surmise that $F(\bar{T})$ is a partition basis for $\bar{T}$. However, we'd be mixing apples and oranges. A basis is necessarily comprised of open sets, not closed sets. The analogue for closed sets would be that every $C \in \bar{T}$ could be expressed as an arbitrary intersection. Here, we are starting with closed sets to obtain a partition via $F(\bar{T})$. However, the elements of that partition are (as arbitrary intersections), closed sets (i.e. $F(\bar{T}) \subseteq \bar{T}$). We can't construct either T or $\bar{T}$ from this by taking arbitrary unions of closed sets. The latter only is closed under finite unions, and the former consists of open sets, not closed sets. On the other hand, if we stuck to open sets, T would not be closed under arbitrary intersections. This means that $F(T) \not\subseteq T$. We get a partition, but not necessarily made of open sets.

A partition topology looks like a discrete topology, but not in the sense of homeomorphism. Rather than the underlying sets S and S' being bijective, only the topologies T and T' are bijective — but in a way which preserves $\cap$ and $\cup$.

Given a partition B of S, we have an equivalence relation $\sim$ on S and an induced topology T_B on S. The quotient set $S' \equiv S/\sim$ is bijective in a natural way with B, so we may as well just substitute B for it. Since any $o \in T_B$ is just a union of sets in B, define a corresponding $u(o) \subseteq B$ to contain the corresponding points in B (ex. if $B = \{(a), (b, c), (d)\}$, then we label $B = \{b_1, b_2, b_3\}$ (where b_1, b_2, and b_3 now are treated as points) and $u((a)) = b_1$, $u((b, c)) = b_2$, $u((d)) = b_3$, $u((a, b, c)) = \{b_1, b_2\}$, etc). Let T' be the collection $T' \equiv \{u(o); o \in T_B\}$. Then T' is a topology on B with exactly the same structure as T_B has on S. Specifically, $u(\cup o_i) = \cup u(o_i)$ for arbitrary unions, and $u(\cap o_i) = \cap u(o_i)$ for finite intersections. Moreover, $B, \emptyset \in T'$. So T' is a topology on B that exactly mirrors the topology T_B. Formally, it is the quotient topology — but it was easier to describe directly in our simple case. Obviously, we cannot have an actual homeomorphism, since S and B aren't bijective. However, T' and T_B 'look' the same in all other regards. Since, by definition, T_B consists of all unions of classes of B, T' consists of all subsets of B. It therefore is the discrete topology on B, and T_B is equivalent to this discrete topology in the sense described.

The topology T_B is blind to any structure below the level of B. I.e., it cannot see the internals of each class. As far as it is concerned, we are working with B and T' as described above. We thus can say that T_B is equivalent to the quotient topology under $\sim$.

2. Morphisms of Partitions of a Set

In mathematics, we're generally more interested in the structure-preserving maps (aka "morphisms") between objects than the objects themselves. Let's consider what the appropriate definition of such morphisms may be in the case of partitions. Clearly, we want a class of maps $f : S \to S'$ that has some sort of property vis-a-vis the classes of whatever partitions B of S and B' of S' are under consideration.

We've already come up with preliminary notions of maps 'to' and 'into' partitions (as well as their 'flexible' counterparts for nonsurjective f's). Let's now take a more comprehensive look at the question of how to construct maps between partitions. We'll tie in those definitions where appropriate, but they won't be our starting point.

2.1. **Motivation in coming up with a definition.** For comparison, consider topology. Given a map $f : S \to S'$ and topologies T on S and T' on S', the relevant morphism is a "continuous map", defined as a map for which $f^{-1}(o') \in T$ for all $o' \in T'$. I.e., the inverse image of an open set is open.

Similarly, given a map $f : S \to S'$ and σ-algebras Σ on S and Σ' on S', the relevant morphism is a "measurable function", defined as a map for which $f^{-1}(s') \in \Sigma$ for all $s' \in \Sigma'$. I.e., the inverse image of a measurable set is measurable.

We will see shortly that the correct analogy of an open set or a measurable set is not a single class of a partition but a union of classes. In fact, we'll see that this is more than an analogy; we, in fact, have a topology and (in the case of a countable partition) a σ-algebra. The morphisms of partitions then are the surjective continuous functions relative to these generated topologies. If B is countable, they are also the measurable functions relative to the generated σ-algebras. We'll return to this important point later. For now, we'll simply take inspiration from topology and measure theory.

Note that the way we engage with partitions differs in an important regard from the usual way topologies and measure spaces are used. In the case of topology and measure theory, we typically (but not always) lock our two objects (T and T' or Σ and Σ') and consider various maps between them. For partitions, we'll often want to lock the map f and consider various pairs of partitions (B of S and B' of S'). This isn't a hard and fast rule, but something to be aware of as we develop the subject. Partitions are lightweight, and there may be many of interest on a given set — even in a particular application. By contrast, although there may be many topologies on a given set, a particular application usually only cares about one of these (per underlying set).

For brevity, we'll often write B and B' (or T and T' or Σ and Σ') without reference to the underlying sets S and S'.

Earlier, we defined maps 'into' partitions and maps 'to' partitions. There, we took a specific, broad approach. We required (i) that f unambiguously induces a map $g : B \to B'$, (ii) that g is surjective, and (iii) that f is surjective.

We argued that (i) is necessary in order to have a meaningful notion of a map from the classes of B to those of B' and is equivalent to requiring that for every $b \in B$, $f(b) \subseteq b'$ for some $b' \in B'$ (what we called the "inclusion condition"), that (ii) is necessary for a meaningful notion of a map from a partition to a partition (as opposed to some subset of its classes), and that (iii) renders our definition stricter but somehow more useful.

Obviously, (iii) implies (ii), but we state them separately here to clarify the reasoning behind each and allow us to relax our assumption to (ii) rather than (iii) in this section.

For the 'forward' definitions (defs 1 - 12) below, we'll need to explicitly require surjectivity of the induced map g. For the 'inverse' definitions (defs 13 - 15b) below, surjectivity of the induced g is automatic once we require a non-empty inverse of each class of B'.

To examine the subject afresh, let's (for now) require surjectivity only of the induced map g, not of f itself. This alone constrains the relative cardinalities of B and B'. It is impossible to have a meaningful map of partitions unless $|B| \geq |B'|$. Otherwise, there will be some class of B' that is not assigned to any class of B. Since B and B' are partitions, $|B| \leq |S|$ and $|B'| \leq |S'|$, which means $|B'| \leq |S|$ as well. It is quite possible for B' to be countable or finite but for B' not to be. However, if B is finite or countable, then B' must be as well.

2.2. **The relationship between f and g.** Before proceeding, let's consider another way of looking at maps in relation to specific partitions. In our earlier discussion of maps 'to' and maps 'into', we focused on maps $f : S \to S'$. We also began our present discussion by motivating such an approach as analogous to that of topology and measure theory. However, there is another way of thinking of maps between partitions — one closer to a literal meaning for that term.

At heart, a map between partitions really is of the form $g : B \to B'$. Given such a map, we can ask which maps $f : S \to S'$ induce it in the way we described above (i.e. $g(b) = [f(b)]$, where $[f(b)]$ is the unique element of b' containing $f(b)$). Once we are given a (surjective) g, all that remains is to pick a function

$h_b : b \to g(b)$ for every b. We'll sometimes refer to these collectively as $h \equiv \{h_b; b \in B\}$. We'll now consider the relationship between f, g, and h.

When we speak of surjectivity (or bijectivity) of the h_b's we'll always mean relative to their relevant b' (i.e. $g(b)$).

If we mention h in the presence of a given g, we'll always mean that the two are compatible, in the sense that $h_b : b \to g(b)$ (and thus Im $h_b \subseteq g(b)$).

Prop 2.1: Given a map $f : S \to S'$ which induces a meaningful map between partitions B and B' in the sense of satisfying the inclusion condition: (i) There is a unique g and h induced by it. (ii) If f is surjective, then g is surjective. (iii) If f is injective, then each h_b is injective. (iv) If f is bijective, then g is surjective and each h_b is injective.

Note that this depends on B and B'. Given B and B', most f's do not induce a map between them, because for some b, $f(b)$ doesn't sit in a single class of B'. Similarly, a given f only induces a map between partitions for some B and B' pairs.

The case where f is surjective corresponds to a 'map into', and the case where we only require g to be surjective corresponds to the broader 'flexible map into'.

Pf: (i) Since we are told that f induces a map between B and B', we know that each $f(b) \subseteq b'$ for some b'. We are not assuming that g is surjective here, of course. By definition, we have a unique g, defined by $g(b) = [f(b)]$. Also by definition, $h_b = f|_b$ is a uniquely defined map from b to $g(b)$. (ii) Suppose f is surjective. Although in aggregate the h_b images must cover S', individually each need not be surjective to its b'. They just have to conspire to fill it in. However, since B' is a partition, each of its classes must have nonempty inverse image. So g must be surjective. (iii) Suppose f is injective. Then every restriction of it is injective too, so all the h_b's are. However, g need not be injective because it still is possible for several disjoint $f(b)$'s to fill in a given b'. (iv) just combines (ii) and (iii).

It is quite possible for f to be surjective but the h_b's not to. After all, each could fill in a part of the relevant b'. It also is quite possible for f to be injective but for g not to be. In fact, even if f is bijective, we could have a non-injective g and non-surjective h_b's. The following example illustrates these situations.

As an example of an f that is bijective but for which g is not injective and some of the h_b's are not surjective, consider $S = S' = \{1, 2, 3, 4\}$ and $B = \{(1), (2), (3), (4)\}$ and $B' = \{(1, 2), (3, 4)\}$ and $f = Id_S$. f is bijective, $g : ((1), (2), (3), (4)) \to ((1, 2), (1, 2), (3, 4), (3, 4))$ is not injective and none of the h_b's ($f|_{(1)} : (1) \to (1, 2)$, etc) is surjective.

The following proposition lets us go the other way as well.

Prop 2.2: Given a map $g : B \to B'$ and a set of maps $\{h_b : b \to g(b); b \in B\}$: (i) there exists a unique $f : S \to S'$ s.t. $f|_b = h_b$ for each $b \in B$, (ii) f flexibly maps B 'into' B' iff g is surjective, (iii) f maps B 'into' B' iff $\cup\text{Im } h_b = S'$, (iv) f flexibly maps B 'to' B' iff g is bijective, and (v) f maps B 'to' B' iff g is bijective and each h_b is surjective (i.e. $g = f'|_B$).

Bear in mind that (iv) does *not* require that f itself be bijective. Each h_b can still be noninjective. There simply cannot be noninjectivity across h_b's.

Pf: (i) Since B is a partition, we just define $f|_b = h_b$ and this is a unique and complete specification. Since each h_b has image in $g(b)$ which is a subset of S', f can legitimately be viewed as a function $S \to S'$. (ii) (forward) Let f flexibly map 'into'. Then each $f(b) \subseteq b'$ for some b', thus defining a unique g, and this g is surjective. In this case $h_b = f|_b$ and we have a (g, h) pair with surjective g. (backward) Given a compatible (h, g) where g is surjective, h_b by definition has Im $h_b \subseteq g(b)$. (iii) (forward) Let f map 'into'. By definition, f is surjective. Since Im $f = \cup\text{Im } h_b$, this equals S'. (backward) Suppose $\cup\text{Im } h_b = S'$. Since Im $h_b \subseteq g(b)$ and $\cup\text{Im } h_b = \text{Im } f$, we meet the conditions of a map 'into'. (iv) (forward) Suppose f flexibly maps 'to'. Then $f(b) \subseteq b'$ and g is bijective. (backward) Suppose g is bijective. Since Im $h_b \subseteq g(b)$ and $f|_b = \text{Im } h_b$, we meet the conditions of flexibly mapping 'to'. (v) (forward) Suppose f maps 'to'. Then $f'(B) = B'$ and $f'|_B$ is bijective and f is surjective. This means $f|_b = g(b)$ (since $g = f'|_B$ in this case), and since $h_b = f|_b$, we have that h_b is surjective to $g(b)$. (backward) Suppose the h_b's are surjective and g is bijective. One and only one h_b has image in a given b'. However, h_b is surjective, so every b' is filled, and f is surjective. We already know that Im $h_b \subseteq b'$, but now we know it is equal to it. Therefore $f|_b = h_b = b'$ and $g = f'|_B$. Since g is bijective, we meet the conditions of a map 'to'.

This raises an interesting question: how much information do we really need in order to deduce the rest? For convenience and to avoid thorny set-theoretic issues, let's assume that all our sets are subsets of some universe U.

Suppose we play a game. Someone gives us a set of functions f_i from disjoint domains S_i to U and asks us to guess S, S', B, B', and f. We can deduce $S = \cup S_i$ and $B = \{S_i\}$ (i.e. the set of domains). We can construct a function $f : S \to U$ via $f|_{S_i} \equiv f_i$ for each i. Since B is a partition, this uniquely and completely defines f. What we don't necessarily know are S' and B'.

Define $S'_0 \equiv \cup \text{Im } f_i$. If we are also told that the relevant f is surjective, then $S' = S'_0$ is the only possibility. However, without this constraint all we can infer is that $S'_0 \subseteq S'$. In either case, we have no way to infer B' from the information given.

For example, B' can *always* be the trivial partition on S'_0. In fact, anything equal to or coarser than $G(f'(B))$ (using our meet-like operation $G()$ from earlier) will work. If we aren't told that f is surjective, then things are even worse. In that case, S' can be any subset of U containing S'_0, and B' can be obtained from any partition of S'_0 equal to or coarser than $G(f'(B))$ by distributing unmapped-to elements of S' between existing classes of that partition (but we *cannot* add any new classes comprised entirely of unmapped-to elements).

Since we cannot infer B' from the set $\{f_i\}$, we require additional information. This is where a compatible g comes in. Im g gives us all we need. We then obtain $S' = \cup_{b' \in \text{Im } g} b'$.

This may seem like too little information. After all, many g's can have the same image. However, there is a crucial implicit assumption as well: compatibility. We've assumed that the given Im g is compatible with the h_b's we were given. This means that $\text{Im } h_b \subseteq b'$ for some $b' \in \text{Im } g$. We then can infer g by associating each b with the b' which its Im h_b inhabits. I.e., we've been told (i) Im g and (ii) that Im g is compatible with the h_b's. Put another way, the only freedom we had was in the choice of unmapped-to space in B'. Given a choice of B', we can infer S', and there is only a single g that is compatible with it and the h_b's.

2.3. **Forward Candidate Definitions.** Let's now construct a list of candidates for our notion of a morphism. We'll begin with 'forward' definitions, in which we constrain the behavior of the $f(b)$'s, and then we'll consider 'inverse' definitions, in which we constrain the behavior of the $f^{-1}(b')$'s (much as we do in topology and measure theory).

As usual, we'll denote by g the induced map $B \to B'$. I.e., each b is taken to the unique b' containing $f(b)$. This may equal $f(b)$ (if $g = f'|_B$) or it may be a subset of it. It will be defined and/or constrained differently in the various definitions.

In enumerating the possibilities, it is useful to consider a visualization. If we think of a given b' as a tray and think of the $f(b)$'s as tiles, we can imagine various ways in which those tiles can cover the tray. We'll visualize overlapping tiles as forming multiple rows. If they overlap wholly (i.e. $f(b_1) = f(b_2)$), then we'll view them as a stack. Otherwise, the rows can be staggered. Our definitions will differ in what sort of tiling behaviors are permissible. Perhaps we allow only one big tile that must completely fill the tray, or perhaps we allow many tiles that may or may not fill the tray but cannot overlap, or perhaps we allow multiple tiles to overlap but only as stacks.

Let's consider several yes/no questions about how the tiles are arranged on the tray:

- (i) Must they completely cover b'?
- (ii) Can there be more than one tile in a given b'?

- (iii) If there are multiple tiles, can they (partly or fully) overlap with one another?

  I.e., do we allow $f(b_1) \cap f(b_2) \neq \emptyset$ if $b_1 \neq b_2$ (subject to the constraint that $g(b_1) = g(b_2)$, of course).

- (iv) If we allow multiple overlapping tiles, must they only overlap fully?

  I.e., if $f(b_1) \cap f(b_2) \neq \emptyset$, do we demand that $f(b_1) = f(b_2)$?

Obviously, these are not unrelated questions. (i) is independent of the rest, but only certain combinations of the other answers make sense. Since g is required to be surjective, every b' must have at least one tile. (i) is tantamount to asking whether f is surjective.

Together, (ii), (iii), and (iv) offer 4 possibilities: we require a single tile, we allow a single row of non-overlapping tiles, we allow stacks of tiles, and we allow partly overlapping tiles. However, we'll find it useful to split one of these into two (we chose four questions to ask but made no claim that they are exhaustive). In the case of disjoint stacks of tiles, we'll distinguish the case with a single stack from that with multiple.

We thus have 5 visual scenarios based on (ii-iv), and two possibilities for each based on (i) (one requiring us to fill b' and one allowing empty space).

Bear in mind that our 'scenarios' revolve around what is permitted, not what actually happens. For example, a single-tile covering of b' satisfies all the scenarios we enumerated. Some scenarios are subsumed by others.

This gives us 10 possibilities, 5 of which involve a surjective f and 5 of which do not. However, two of the surjective cases (the single big tile filling b' and the single row of ≥ 1 disjoint tiles filling b') admit the possibility that f could be bijective. We thus add two more candidates which replicate these scenarios but also require f to be bijective rather than merely surjective. By this means, we find ourselves with 12 total candidates: 5 with a nonsurjective f, 5 with a surjective f, and 2 with a bijective f. Diagram 1 illustrates these scenarios and their connection with our definitions below.

- Def 1: The most permissive choice is to (i) require that each $f(b)$ reside in one and only one b' (i.e. our "inclusion condition" from earlier), (ii) define $g(b)$ to be this b', and (iii) require that g be surjective to B'.
- Def 2: We can tighten def 1 and require that the $f(b)$'s that map to a given b cover that b'. This is tantamount to requiring that f be surjective. Each b' is filled by images of classes of B, and there are no constraints on how these images behave within a given b'. I.e., each b' is covered in tiles that can overlap in whole or in part.
- Def 3: We can tighten def 1 and require that the $f(b)$'s be disjoint unless they are equal. I.e., they cannot partly overlap, and $f(b_1) = f(b_2)$ or $f(b_1) \cap f(b_2) = \emptyset$. This is tantamount to requiring that $f'|_B \subseteq B'_R$ for some refinement B'_R of B' and that g is surjective to B'. Each b' is filled wholly or in part by tiles that can stack but cannot partly overlap.

  Note that there can be more than one choice of B'_R if f is non-surjective. Also note that we are requiring g to be surjective but not necessarily injective. Two classes of B can map to the same $f(b)$ (and thus form stacked tiles). Finally, note that $g \neq f'|_B$. g is from B to B' and $f'|_B$ is from B to B'_R.

- Def 4: We can tighten def 3 and require that the $f(b)$'s be distinct. I.e., $f(b_1) \cap f(b_2) = \emptyset$ if $b_1 \neq b_2$. Each b' now is filled in whole or in part by unstacked disjoint tiles.

  Again, there can be more than one choice of B'_R if f is non-surjective.

- Def 5: We can tighten def 3 to allow only one stack (that need not fill b'). This is tantamount to requiring that $g = f'|_B$ and g is surjective and each $f(b_1) = f(b_2)$ or $f(b_1) \cap f(b_2) = \emptyset$ and that each b' contains only one such stack (i.e. $\cup f(b_i) = f(b_j)$ for any j in the union).
- Def 6: We can tighten either def 5 to prohibit stacking or def 4 to require that no two classes of b map to the same b' (i.e. that g be bijective). These give the same result, and each b' is now filled (in whole or in part) by a single tile.
- Def 7: We can tighten either def 2 to prohibit partial overlaps or def 3 to require that each b' be covered by (possibly stacked) tiles. This is tantamount to requiring that $f'|_B = B'_R$ for some refinement B'_R of B' and that f is surjective. Each b' now is completely covered by disjoint (but possibly stacked) tiles.

  Surjectivity of f implies surjectivity of g. Note that there is now only one choice of refinement B'_R that fills the requisite role.
- Def 8: We can do one of three equivalent things: (i) tighten def 2 to prohibit overlapping tiles, (ii) tighten def 4 to require that the (unstacked) tiles fill b', or (iii) tighten def 7 to prohibit stacking of tiles. All three yield the same result and are tantamount to requiring that f be surjective, that $f|_B = B'_R$ for some refinement B'_R of B', and that g be bijective. Each b' is now covered with disjoint unstacked tiles.

  Surjectivity of f implies surjectivity of g, so we're only additionally requiring that g be injective. As in def 7, there is now a single choice of refinement B'_R that works.
- Def 9: We can tighten either def 7 to require that each $f(b)$ equal a whole class of B' or def 5 to require that the single stack fill b'. This is tantamount to requiring that $g = f'|_B$ and f is surjective. Each class of B now maps to a complete class of B', but we still can have stacking. Each b' is covered by a single stack.
- Def 10: We can do one of three equivalent things: (i) tighten def 9 to disallow stacking or (ii) tighten def 8 to require a single tile per b' or (iii) tighten def 6 to require that the single tile fills b'. All are tantamount to requiring that f be surjective and $g = f'|_B$ and g is bijective. Each b' is covered by a single unstacked tile. However, f need not be bijective. It can be non-injective within each b.
- Def 11: We can tighten def 8 to require that f be bijective. Each b' is still covered by disjoint unstacked tiles as in def 8, but now we disallow any noninjectivity of f.
- Def 12: We can tighten either def 11 to allow only one tile per b' or def 10 to require that f be bijective. Both are tantamount to requiring that f be bijective, $g = f'|_B$, and g is surjective (which implies that it is bijective in this case). As in def 10, we have a single unstacked tile covering each b'.

As can be seen directly from these definitions:

- Def 1 corresponds to our 'flexible map into'.
- Def 2 corresponds to our 'map into'.
- Def 6 corresponds to our 'flexible map to'.
- Def 10 corresponds to our 'map to'.

2.4. **Inverse Candidate Definitions.** So far, our definitions have been forward facing. We've constrained the behavior of the $f(b)$ tiles. As mentioned, we can also take a cue from topology and measure theory and try to define morphisms in terms of the behavior of inverses.

We know from proposition 1.7 that the pullback f^*B' (i.e .$\{f^{-1}(b'); b' \in B'\}$) is a partition of S, and from proposition 1.8 we know that if f is surjective then it maps f^*B' 'to' B'. However, there need exist no relationship between f^*B' and whatever B we are interested in. In general, f^*B' can be either (i) a refinement of B, (ii) a coarsening of B, (iii) equal to B, or (iv) none of the above. Since any notion of morphism we come up with can't brook ambiguity and must assign a single class of B' to each class of B, only (ii) and (iii) are viable. In the other cases, we do not have an induced function $B \to B'$.

- Def 13: We can require that $f^{-1}(b')$ be a class of B for every $b' \in B'$, and define our g as $g(b) = [f(b)]$ (where, as before, $[f(b)]$ is the unique b' s.t. $f(b) \subseteq b'$). We are guaranteed that g is well-defined and bijective. However, f need not be surjective. Each b' is wholly are partly filled with a single tile, corresponding to def 6.

  Pf: (forward) Suppose def 13 holds. To satisfy def 6, we need each $f(b) \subseteq b'$ for some b' (thus defining $g(b) = [f(b)]$) and for this g to be bijective. Pick $b \in B$. Since B' is a partition, $f(b)$ must intersect at least one b'. Moreover, $b \subseteq f^{-1}(f(b))$ from set theory. By def 13, $f^{-1}(b') = b_1$ for some b_1. Therefore $b \cap b_1 \neq \emptyset$. Since B is a partition, $b = b_1$. This shows that $f(b) \subseteq b'$ and also that $b \in f^*B'$. I.e., $B \subseteq f^*B'$. However, since both are partitions, they must be equal (or B wouldn't cover S). So $B = f^*B'$. Earlier, we saw an example where a non-surjective f doesn't map f^*B' 'into' B'. The problem there was that $f^{-1}(b')$ was empty for some b'. However, we've proscribed that here by requiring in def 13 that every $f^{-1}(b')$ be a class of B and thus nonempty. I.e., f induces a map g from f^*B' to B' that is surjective and thus (by construction) bijective. We therefore have seen that, in order to satisfy def 13, $B = f^*B'$ and the induced g is bijective. We now have satisfied def 6 and thus have a flexible map 'to'. (backward) Now, suppose def 6 holds. For every b, there is a b' s.t. $f(b) \subseteq b'$ and the resulting g is bijective. This means there is one and only one b which maps into a given b', and every b' has such a b. Since B is a partition, $f^{-1}(b')$ must equal the unique b with $f(b) \subseteq b'$. Therefore def 13 holds.

- Def 13a: We can tighten def 13 to require that f be surjective. This corresponds to def 10.
- Def 13b: We can tighten def 13a to require that f be bijective. This corresponds to def 12.
- Def 14: We can require that $f^{-1}(b')$ be a nonempty union of classes of B for every $b' \in B'$. This means that each $f(b)$ sits in a specific b', but more than one $f(b)$ can sit in the same b'. We are still guaranteed that g is well-defined and surjective, but it need no longer be injective. We've imposed no constraints on how the tiles behave inside of a given b', so this corresponds to def 1.

  Pf: (forward) Suppose def 14 holds. To satisfy def 1, we need that every $f(b) \subseteq b'$ for some b' and that the resulting g is surjective. Consider some $b \in B$. Since B' is a partition, $f(b)$ must intersect at least one b'. Moreover, $b \subseteq f^{-1}(f(b))$ from set theory. By def 14, $f^{-1}(b') = \cup b_i$ for some nonempty union. Therefore $b \cap \cup b_i \neq \emptyset$. Since B is a partition, this can only happen if $b = b_i$ for some b_i. I.e., $b \subseteq f^{-1}(b')$. This means that $f(b) \subseteq f(f^{-1}(b')) \subseteq b'$. So $f(b) \subseteq b'$. By construction, the induced g is surjective, so def 1 holds. (backward) Suppose def 1 holds. Consider some b'. Since g is surjective, every b' is mapped to from some b. Therefore, $f^{-1}(b') \neq \emptyset$. Since B is a partition, $f^{-1}(b')$ must intersect at least one b. By def 1, each b has $f(b) \subseteq b'_1$ for some b'_1. Since $f(b) \cap b' \neq \emptyset$, this b'_1 must be b'. I.e., $f(b) \subseteq b'$. Since this is true for any b that has $f(b) \cap b' \neq \emptyset$, $f^{-1}(b')$ is a union of classes of B. Def 14 therefore holds.

- Def 14a: We can tighten def 14 to require that f be surjective. This corresponds to def 2.
- Def 14b: We can tighten def 14a to require that f be bijective. This corresponds to def 11.

Any other possibility for $f^{-1}(b')$ would result in an $f(b)$ that spans more than one class of B' and thus fails to induce a meaningful notion of morphism. Note that defs 13 and 14 (and all their variants) guarantee that the induced map g is surjective to B'.

Since defs 13 and 14 (in all their variations) are the only valid inverse-based definitions, and they correspond to defs 1, 2, 6, 10, 11, and 12, what about defs 3, 4, 5, 7, 8, and 9? These seemed reasonable, so shouldn't they have counterparts too? All of these are derived from (and special cases of) one or more of the forward-definitions which correspond to our inverse definitions. As such, we could contrive a corresponding inverse definition for these derivative definitions. However, the exercise isn't particularly illuminating.

How does this compare with topology and measure theory? An obvious first guess would be to view classes of a partition as the analogs of open sets or measurable sets. We then would posit def 13 (or def 13a if we're sticking to surjective f's) as the counterpart of a continuous or measurable function. Instead of the inverse image of an open set being open, the inverse image of a class is a class. However, on closer thought, this doesn't quite fit. Defs 13 and 13a involve an induced bijection g between partitions, which looks more like an isomorphism than a morphism.

Where did we go wrong? The problem is that an open set in topology is a broader concept than a class in partition theory. Unions of open sets are open sets, as are finite intersections. There are no nontrivial intersections of classes of a partition, but we still can look at unions. To do so, we'll need to consider maps $2^B \to 2^{B'}$. The counterpart of continuity/measurability then becomes:

- Def 15: We require that $f^{-1}(\cup b'_i) = \cup b_j$ be a nonempty union of classes of B for any union of classes of B'.
- Def 15a: We can tighten def 14 to require that f be surjective.
- Def 15b: We can tighten def 14a to require that f be bijective.

As the following proposition tells us, this turns out to be no more general than def 14. I.e., instead of looking to def 13 for our analogue of continuity/measurability, we simply needed def 14.

Prop 2.3: Def 15 is equivalent to def 14 (and thus to def 1).

Pf: (backward) Suppose def 14 holds. From basic set theory $f^{-1}(\cup b'_i) = \cup f^{-1}(b'_i)$, but by def 14 each $f^{-1}(b'_i)$ is a union of classes of B. Therefore, 15 is satisfied. (forward) Suppose def 15 holds. Then $f^{-1}(\cup b'_i) = \cup b_j$ (and the union is nonempty) for each nonempty $\cup b'_i$. It may seem that this is more general and could accommodate cases where a b_j maps to more than one b'. However, this is not the case. A single set union still is a union, so $f^{-1}(b') = \cup b_j$ for some nonempty union, and def 14 holds. Where did our perceived extra generality fail? Suppose $f(b)$ is split between classes b'_1 and b'_2. Then $f^{-1}(b'_1)$ isn't a union of classes of B because it contains only a fragment of b.

Note that if B is countable, then we end up with a countable union in the proof.

The correct analogue of continuity/measurability is now clear. Defs 1/14 (or 2/14a if we're sticking with surjective f's) fit the bill nicely. These will, in fact, constitute our notion of a morphism between partitions.

What about the analogue of homeomorphism in topology or bimeasurability in measure theory? In those cases, we need f to be bijective with continuous or measurable inverse. There are only two candidates that involve a bijective f, and only one of these has a bijective g as well. Unsurprisingly, def 12/13b does the trick and will constitute our notion of 'isomorphism'.

Since we now have identified the correct notions of morphism and isomorphism, let's state them formally:

- A **morphism** of partitions is def 1/14/15.

  I.e., it is a 'flexible map into'.

- A **surjective morphism** of partitions is def 2/14a/15a.

  I.e., it is a 'map into'.

- An **isomorphism** of partitions is def 12/13b.

  I.e., it is a 'map to' for which f is bijective.

Before proceeding, let's tie this definition of isomorphism to the usual notion of an invertible morphism.

Prop 2.4: f is an isomorphism of partitions iff f is bijective, a morphism, and f^{-1} is a morphism.

Obviously, if f is bijective, then f and f^{-1} can only be surjective morphisms, not plain morphisms.

Pf: To satisfy def 12, we need f to be bijective, $g = f'|_B$, and g to be surjective. To satisfy def 2, we need f to be surjective, every $f(b) \subseteq b'$ for some b', and the resulting g to be surjective. (forward) Suppose f is an isomorphism. By def 12, f is bijective. Therefore, proposition 1.1 tells us that $f'|_B$ is bijective to its image. Since $g = f'|_B$ is surjective, $f'(B) = B'$, and g is bijective to B'. This patently satisfies def 2, since $f(b) = b'$ for some b' and g is surjective. In the other direction, proposition 1.1 tells us that $(f^{-1})' = f'^{-1}$ since f is bijective. Therefore, g^{-1} is the induced map on the way back. f^{-1} is surjective, $f^{-1}(b') = b$ since g is bijective with inverse $(f^{-1})'|_{B'}$, and the induced map g^{-1} is surjective. The inverse therefore is a surjective morphism. (backward) Suppose f is bijective, a morphism, and f^{-1} is a morphism. As a morphism, each $f(b) \subseteq b'$ for some b'. Since f^{-1} is a morphism, $f^{-1}(b') \subseteq b_1$ for some b_1. Since $b \subseteq f^{-1}(f(b)) \subseteq f^{-1}(b') \subseteq b_1$, $b \subseteq b_1$. However, B is a partition, so $b = b_1$. I.e., the induced map of f^{-1} is just g^{-1}. Therefore, g must be bijective. Since f is bijective, $f(b)$ automatically is injective to b'. Suppose that $f(b) \neq b'$. Then for some $x' \in b'$ there is no $x \in b$ that maps to it. However, g is bijective, so no other $f(b_1)$ can map to it. This means f is non-surjective, violating our assumption. So $f(b) = b'$ and $g = f'|_B$. We thus satisfy def 12.

Table 1 compares our various definitions and their properties. Diagram 1 provides a visualization of their behavior and relationships.

2.5. **Upgrading the Topological and Measure-Theory Analogies.** It turns out that we can upgrade the topological/measure-theory analogy to an actual equality. I.e., rather than merely comparing our notions with those of topology and measure theory, we can actually work (to some degree) *with* topology and measure theory — and not just any topology or measure theory, but the simplest of all: discrete topologies and power-set σ-algebras.

Since B and B' are partitions of S and S', we can use them to generate topologies on S and S'. Call these T and T'. A partition basis is also a subbasis for a topology, so it doesn't matter whether we generate T using B as a basis or subbasis. The open sets of T are unions of classes of B.

Since T and T' have partition bases, they are quotient-equivalent to discrete topologies as discussed earlier. Denote these quotient topologies T_B and $T'_{B'}$.

Recall that T is not homeomorphic to T_B and T' is not homeomorphic to $T'_{B'}$. They are merely quotient-equivalent.

If B is countable, the requirement that g (but not necessarily f) is surjective means that B' must be countable as well. A countable partition can serve as a basis (or, equivalently, subbasis) for a σ-algebra. Let Σ on S and Σ' on S' be the generated σ-algebras.

As with T and T', we can define quotient-equivalent σ-algebras that have B and B' as their sets of elementary events. These are powerset σ-algebras, and we'll denote them Σ_B and $\Sigma'_{B'}$.

The Σ generated by B is just the Borel algebra of the T generated by B. In fact, $T = \Sigma$ and $T' = \Sigma'$. The measurable functions from B to B' are the continuous functions from T to T'.

Although a topology allows arbitrary unions in its definition, when B is countable this is irrelevant. Every element of T is a countable union of elements of B (since B is countable), so its complement is too.

It is easy to see that $T_B = \Sigma_B$ and $T'_{B'} = \Sigma'_{B'}$ as well.

Given general topologies T and T' on sets S and S', suppose Σ and Σ' are their Borel algebras. In general, there are more measurable functions between the Borel algebras than continuous functions between the topologies. However, if T is generated by a countable partition basis, then its Borel algebra also is generated by that same partition basis. But how can this be? The unions of basis sets are open in T, but Σ contains closed sets too (as well as countable unions of open and closed sets). If each of these is expressible as a union of basis elements, then they must be open. The answer is that they are. Every set in T is clopen. In fact, it is easy to see that $T = \Sigma$. The complement of a union of basis elements is a union of the remaining basis elements — and hence open. Therefore, every element of T is clopen. This is no surprise and is why T is quotient-equivalent to the discrete topology. In summary, for a countable basis B, the generated T and Σ satisfy $T = \Sigma$, and T is its own Borel algebra. Consequently, every measurable function is continuous.

The following proposition tells us the relationships between our morphisms/isomorphisms and continuous functions/homeomorphisms and measurable functions/bimeasurable functions.

Prop 2.5: Let B and B' be partitions of S and S', let T and T' be the topologies on S and S' generated by B and B', and let T_B and $T'_{B'}$ be the corresponding discrete quotient topologies. If B is countable, let Σ and Σ' be the σ-algebras on S and S' generated by B and B' and let Σ_B and $\Sigma_{B'}$ be the corresponding powerset quotient σ-algebras. Let $f : S \to S'$. Then:

As mentioned, since B and B' are partitions, it doesn't matter whether we consider them subbases or bases to the topologies and (in the countable case) σ-algebras.

- (i) f is a surjective morphism from B to B' iff f is a surjective continuous function from T to T' iff f is a surjective continuous function from T_B to $T'_{B'}$.
- (ii) If f is a morphism from B to B' then f is a continuous function from T to T' and f is a continuous function from T_B to $T'_{B'}$. The converse need not hold.
- (iii) f is an isomorphism between B and B' iff f is a homeomorphism between T and T' iff f is a homeomorphism between T_B and $T'_{B'}$.
- (iv) For B countable, f is a surjective morphism from B to B' iff f is a surjective measurable function from Σ to Σ' iff f is a surjective measurable function from Σ_B to $\Sigma'_{B'}$.
- (v) For B countable, if f is a morphism from B to B' then f is a measurable function from Σ to Σ' and f is a measurable function from Σ_B to $\Sigma'_{B'}$. The converse need not hold.
- (vi) For B countable, f is an isomorphism between B and B' iff f is bimeasurable between Σ and Σ' iff f is bimeasurable between Σ_B and $\Sigma'_{B'}$.

We won't prove the equivalence between maps involving T and T' and those involving T_B and $T_{B'}$ (and ditto for Σ, etc). Those are standard topology and measure-theory results. We also saw that when B is countable — in our special case — there is an equivalence between measurable fns and continuous functions. This extends to an equivalence between homeomorphisms and bimeasurable functions (trivially, since the inverse-image defs are the same). Therefore, we won't trouble with all these. We'll just focus on proving the equivalence between morphisms/isomorphisms from B to B' and continuous functions/homeomorphisms from T to T'. I.e., we'll only prove the key parts of (i)-(iii), and the rest follows trivially.

Pf: (i) The unions of classes are the open sets, so it may seem like $f^{-1}(o') \in T$ iff $f^{-1}(\cup b'_i) = \cup b_j$. However, we must be careful. The definition of continuity allows $f^{-1}(o') = \emptyset$, but def 15 does not. In the case of (i), f is surjective, and we are guaranteed that $f^{-1}(\cup b'_i) \neq \emptyset$ for any nonempty $\cup b'_i$. Therefore, the defs are the same.

Pf: (ii) Now we run into the problem warned of in our proof of (i). If $f^{-1}(\cup b'_i) = \cup b_j$ for all unions, then $f^{-1}(o') \in T$ for all $o' \in T'$. However, the converse need not hold. Suppose f is continuous, so $f^{-1}(o') \in T$ for all $o' \in T'$. If $f^{-1}(o') = \emptyset$, then $f^{-1}(\cup b'_i = o') = \emptyset$ for some nonempty $\cup b'_i$, and we fail to produce a morphism.

(ii) converse counterexample: Let $S = \{1, 2\}$ and $S' = \{1, 2, 3, 4\}$ and $B = \{(1), (2)\}$ and $B' = \{(1), (2), (3), (4)\}$ and $f : (1, 2) \to (1, 2)$. Then T and T' are the discrete topologies on S and S', and f is continuous between them — since *any* function from the discrete topology is continuous relative to it. However, f cannot be a morphism from B to B', because the induced g is non-surjective. $f^{-1}((3)) = f^{-1}((4)) = \emptyset$.

Pf: (iii) (forward) Suppose f is a homeomorphism. Then it is bijective, continuous, and has continuous inverse. This means it is surjective and continuous in each direction. However, from (i) we know that surjective and continuous implies a surjective morphism. So we have a bijective f that is a surjective morphism in each direction. (ii) (backward) Suppose f is an isomorphism. From proposition 2.4, it is bijective, and a morphism in each direction. This means it is a surjective morphism in each direction. From (i), we know that a surjective morphism is a surjective continuous function. So, f is bijective, continuous, and has continuous inverse — making it a homeomorphism.

Pf: (iv-vi): Since we proved this for continuous functions/homeomorphisms, it immediately follows for measurable/bimeasurable functions when the partitions are countable. As mentioned in an earlier comment, the continuous and measurable functions are the same in the specific case of the topology and σ-algebra generated from the same countable partition.

What about the 'missing' definitions? We've identified morphisms and surjective morphisms and isomorphisms with their topological and measure-theory equivalents, but what about the remaining 9 definitions? These just correspond to particular continuous (or surjective continuous) functions, and do not necessarily have named topological or measure-theory counterparts.

Without going into the details, we'll simply observe that:

- 11, 14b, 15b describe a topological quotient-equivalence (similar to, but distinct from, the discrete quotient equivalence discussed earlier). Denoting by $\sim$ the equivalence relation on S defined by partition B, and letting $T/\sim$ be the corresponding quotient topology on S, $T/\sim$ is homeomorphic to T', and $T/\sim$ is a coarsening of T.

  Equivalently, T is homeomorphic to a "supertopology" of T', by which we mean the opposite of a subtopology. I.e., T is homeomorphic to some T'' s.t. $T' \subset T''$. Note that we are *not* talking about changing S'. Both T' and T'' are topologies on S', but T'' has more open sets. We can think of it as "refining" T'.

- Despite appearances, 6, 13 do *not* describe a (possibly nonsurjective) continuous open map.

  Recall that an open map takes open sets to open sets. I.e., $f(o) \in T'$ for all $o \in T$. To be an 'open map', we would need $f(\cup b_i) = \cup b'_j$. Restricting to single-class unions on the left, we have that each $f(b) = \cup b'_j$ for some union. However, each $f(b) \subseteq b'$ for a single b', so the union on the right can only have a single element. I.e., we need each $f(b) = b'$ for some b'. This is tantamount to $g = f'|_B$.

- 3 describes a (possibly nonsurjective) continuous function to a supertopology of T'.
- 5 is just a special case of 3 that we chose to separate out.
- Despite appearances, 4 does *not* describe an injective (but possibly nonsurjective) continuous function to a supertopology of T'.

  Although the induced map of open sets would be injective, f itself need not be. There can be non-injectivity internal to each $f(b)$.

- 7 describes a surjective continuous function to a supertopology of T'.
- Note that 8 is just a special case of 7. f need not be injective, even though g is.
- 9 describes a surjective continuous open map.

  However, not all open maps between the generated topologies arise this way. Since an open set is a union of our partition classes, an open map takes unions of classes to unions of classes. Therefore, it must take each class to a union of classes. However, to be a morphism (which every definition is, since they all are tightenings of 1, 14, 15), a class can only map to a single other class. I.e., only a restricted subset of the open maps are morphisms.

  Note that none of our definitions corresponds to a nonsurjective open map, because that would require that at least one whole class of B' be absent from the image — thus rendering g nonsurjective.

- 10, 13a is just a special case of 9. f need not be injective, even though g is.

The following propositions regarding isomorphisms will prove useful.

Prop 2.6: Let $f : S \to S'$ be bijective, and let B be a partition of S and B' be a partition of S'. (i) f is an isomorphism from B to $f'(B)$ and from B to no other partition and (ii) f is an isomorphism from f^*B' to B' and from no other partition to B'.

I.e., f cannot act as an isomorphism to or from two distinct partitions if we hold the other end fixed.

Pf: (i) We'll use def 12. To be an isomorphism, f must be bijective. From proposition 1.1, we know that $f' : 2^S \to 2^{S'}$ is bijective as well. Since it is injective, its restriction is injective, so $f'|_B$ is injective. Since we defined $B' = f'(B)$, we automatically have $g = f'|_B$, which is surjective to B' since f is surjective to S'. However, it also is injective as a restriction of f'. Therefore, the induced g is bijective to B' and f is an isomorphism. Now, suppose that f is an isomorphism from B to some other B''. Then each $f(b) = b''$ for some unique $b'' \in B''$ (and the resulting g is bijective). However, we already know that $f(b) = b'$ for some $b' \in B'$. Along with the bijectivity of both induced g's, this gives us that $B'' = B'$.

Pf: (ii) Let $B = f^*B'$. Then $g = f'|_B$ and is bijective, by construction. We can use the argument from (i) but for f^{-1} to establish uniqueness.

Prop 2.7: Given $f : S \to S'$, the following conditions are equivalent: (i) f is bijective, (ii) for every partition B of S, there exists a unique B' on S' s.t. f is an isomorphism from B to B', (iii) for every partition B' of S' there exists a unique B of S s.t. f is an isomorphism from B to B', (iv) for every partition B' of S', there exists a unique B on S s.t. f^{-1} is an isomorphism from B' to B, (v) for every partition B of S, there exists a unique B' of S' s.t. f^{-1} is an isomorphism from B' to B.

Pf: (ii→i,iii→i,iv→i,v→i): If there exists any B and B' for which f or f^{-1} is an isomorphism, then f must be bijective. Uniqueness follows from proposition 2.6.

Pf: (i→ii): Let f be bijective. Define $B' = f'(B)$. Then $g = f'|_B$ is a bijection to B', and f is an isomorphism from B to B'.

Pf: (i→iii): Let f be bijective. Define $B = f^*B'$. Then g is patently bijective to B' and equals $f'|_B$. Therefore, f is an isomorphism from B to B'.

Pf: (i→iv): From (iii), we know there is a unique B s.t. f is an isomorphism from B to B'. This means f^{-1} is an isomorphism from B' to B.

Pf: (i→v): From (ii), we know there is a unique B' s.t. f is an isomorphism from B to B'. This means f^{-1} is an isomorphism from B' to B.

Prop 2.8: The inverse of an isomorphism is an isomorphism.

Pf: Let $f : S \to S'$ be an isomorphism from B to B'. Proposition 2.4 tells us that f is bijective, a morphism, and its inverse is a morphism. This is symmetric. f has these properties iff f^{-1} does too.

Prop 2.9: (i) Isomorphisms compose to an isomorphism. (ii) Two surjective morphisms compose to an isomorphism iff they are isomorphisms.

I.e., we can't mix an isomorphism and a nonisomorphism bijective morphism.

Pf: (i) Let f_1 and f_2 both be isomorphisms. By proposition 2.4, this means that each is bijective, a morphism, and has an inverse that is a morphism. The composition of bijections is a bijection, so $f_2 \circ f_1$ is bijective. A bijective morphism is a surjective morphism, aka a 'map into'. By proposition 1.5, these compose. So $f_2 \circ f_1$ is a surjective morphism from B_1 to B_3. Similarly, $(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$ is a composition of surjective morphisms and thus a surjective morphism. We therefore see that $f_2 \circ f_1$ is an isomorphism.

Pf: (ii) We already have one direction from (i). Suppose f_1 and f_2 are surjective morphisms. (bijectivity) Since $f_2 \circ f_1$ is a bijection, f_1 is injective and f_2 is surjective. We already know the latter, but we also know that (as a surjective morphism), f_1 is surjective. Therefore f_1 is bijective. This means that f_2 must be a bijection as well. We thus have two bijective morphisms. There do exist nonisomorphism bijective morphisms (we'll later call these 'coarsening morphisms'). Let's first show that one of these composed with an isomorphism doesn't give an isomorphism (i.e. we can't mix types). Suppose f_1 is an isomorphism and $f_2 \circ f_1$ is an isomorphism but f_2 is not. Since f_1 and f_2 are bijective, we can speak of their inverses. In particular, we can write $(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$. By proposition 2.8, f_1^{-1} is an isomorphism. By part (i) of the present proposition, $f_1 \circ (f_1^{-1} \circ f_2^{-1})$ is an isomorphism. But this equals f_2^{-1}. Since f_2 is an isomorphism iff f_2^{-1} is one, this violates our premise. We thus cannot compose an isomorphism with a nonisomorphism bijective morphism to get an isomorphism. Now, let's show that we can't compose two nonisomorphism bijective morphisms to get an isomorphism. (can't create iso from two noniso) Consider $b_1 \in B_1$. We know that f_1 is a morphism, so $f_1(b_1) \subseteq b_2$ for some $b_2 \in B_2$. Similarly, f_2 is a morphism so $f_2(b_2) \subseteq b_3$ for some $b_3 \in B_3$. This means that $f_2(f_1(b_1)) \subseteq b_3$. However, we know that $f_2(f_1(b_1)) = b_3'$ for some $b_3' \in B_3$ since $f_2 \circ f_1$ is an isomorphism. B_3 is a partition, so we must have $b_3 = b_3'$. Since $f_2 \circ f_1$ is injective, its restriction to b_1 must be bijective to its image b_3. Let $s_2 \equiv f_1(b_1)$. We know that $s_2 \subseteq b_2$ and $f_2(b_2) \subseteq b_3$. If $f_2(b_2) \neq b_3$, then it is impossible for $f_2(f_1(b_1)) = b_3$. f_2 therefore is surjective to b_3. Since f_2 is bijective (and thus injective), any restriction of it is injective. So $f_2|_{s_2}$ is bijective to b_3. Suppose $s_2 \neq b_2$. Then some $y' \in b_2$ isn't mapped to by $f_1|_{b_1}$. However, since f_2 is injective, $f_2(y')$ can't be mapped to from any other element of b_2. I.e., $f_2(y')$ isn't in the image of $f_2 \circ f_1$, contradicting our earlier result that $f_2 \circ f_1$ is bijective from b_1 to b_3. Therefore, $f_1|_{b_1}$ is surjective to b_2. It is injective, so this makes it bijective to b_2. We therefore know that the induced $g_1 = f_1'|_{B_1}$ and $g_2 = f_2'|_{B_2}$. We also know that both are surjective. However, by proposition 1.1 we know that f_1' and f_2' are injective. This makes them bijective. We have bijective f's and g's, so f_1 and f_2 are isomorphisms.

Note that the converse does *not* hold if f_1 and f_2 are not morphisms. Morphisms force us in a certain direction. General functions don't have this problem. For example, let $S_3 = S_1$ and $B_3 = B_1$. If $f_2 \circ f_1 = Id_{S_1}$, then it is an isomorphism from B_1 to B_3 (i.e. an automorphism of B_1). Let f be any bijective function from S_1 to S_2. Then f and f^{-1} compose to Id_{S_1}. However, f need not be a morphism from B_1 to B_2, and f^{-1} need not be a morphism from B_2 to B_1.

2.6. **Coarsening and Refining Morphisms.** We know what refinement and coarsening look like for partitions of a given S, and proposition 1.3 offers us a relationship between the partitions mapped 'to' and 'into' by a given B as well as those which map 'to' and 'into' a given B'. We can use the notion of morphisms and isomorphisms to extend this to partitions of a different S', while at the same time gaining some insight into what certain morphisms look like.

Consider the act of coarse-graining. Given two partitions B and B' on S, how do we know whether B is a refinement of B'? There is an easy test.

Prop 2.10: Let B and B' be partitions of S. B is a refinement of B' (proper or not) iff $f = Id_S$ is a surjective morphism from B to B'.

Pf: (backward) Suppose Id_S is a surjective morphism. As a morphism, each $f(b) \subseteq b'$ for some b'. Since $Id_S(b) = b$, this means each $b \subseteq b'$ for some b'. Since f is surjective, each b' is a union of classes of B. Since B and B' are both partitions, B therefore is a refinement of B'. (forward) Suppose B is a refinement of B'. Then $Id_S(b) = b \subseteq b'$ for some b'. So we have a morphism. Id_S patently is surjective, which means the induced g is surjective too. I.e., we have a surjective morphism.

Note that we can't go in the opposite direction unless we have equality. To go from a coarser partition B' to a proper refinement B would involve splitting a single class of B' across several classes of B, a no-no when it comes to partition maps of any sort.

We don't have a counterpart of Id_S when comparing partitions of different sets, but there's an obvious way to leverage the notion. From our standpoint, isomorphic partitions are the same. Given (S, B) and (S', B'), what does it mean for B' to be a "refinement" of B? We first perform a refinement of (S, B) to some (S, B_R), and then identify an isomorphism from (S, B_R) to (S', B').

A coarsening should look like a merger of classes and a refinement should involve splitting classes into disjoint covers. To generalize these notions, we'll at least need a bijective f. If f isn't surjective then we'll have gaps in B', and if it isn't injective we can have overlaps.

Noninjectivity of f need not translate into overlaps (i.e. a noninjective g to the relevant coarsening or from the relevant refinement), but that's not the issue here. A refinement should look like a simple splitting (i.e. like a generalization of our identity map) and a coarsening should look like a simple merger. Noninjectivity of f, even if harmless from the standpoint of partition maps, still would fail to represent such notions. We'll therefore require f to be bijective when it comes to generalizing the notions of refinement and coarsening. We certainly could generalize even further to allow such intra-$f(b)$ noninjectivity — and this would amount to replacing "isomorphism" in the definitions below with "map to" — but this accomplishes little more than giving a name to a particular set of scenarios (specifically, to def 8 as opposed to def 11). It reflects a slightly different notion, and one we have no good reason to entertain here.

Given a bijective $f : S \to S'$, there are four possibilities we'll consider:

- f is an isomorphism from B to a coarsening of B'. We'll term f an **uncoarsening nonmorphism**.
- f is an isomorphism from B to a refinement of B'. We'll term f a **coarsening morphism**.
- f is an isomorphism from a coarsening of B to B'. We'll term f a **refining morphism**.
- f is an isomorphism from a refinement of B to B'. We'll term f an **unrefining nonmorphism**.

The nomenclature is meant to emphasize that, unless it's an isomorphism, f is not even a morphism from B to B' in the uncoarsening/unrefining cases. We will see shortly that uncoarsening/unrefining nonmorphisms are the same thing and coarsening/refining morphisms are the same thing.

Whether or not we include isomorphisms in these four definitions is a matter of choice. We'll include them, which will prove convenient in some ways and inconvenient in others. When we wish to exclude isomorphisms, we'll speak of nonisomorphism coarsening morphisms, nonisomorphism unrefining nonmorphisms, etc. In that case, the relevant coarsenings and refinements in the definitions are proper.

We also can consider four more general cases:

- (a) f is an isomorphism from a coarsening of B to a coarsening of B'.
- (b) f is an isomorphism from a refinement of B to a coarsening of B'.
- (c) f is an isomorphism from a coarsening of B to a refinement of B'.
- (d) f is an isomorphism from a refinement of B to a refinement of B'.

Cases (a) and (d) are of no interest to us, because we're considering only a bijective f. Any bijective f is trivially an isomorphism between the singleton partitions of S and S', which are refinements of all partitions on those sets. I.e. given any bijective f and any B and any B', f is an isomorphism from some refinement of B to some refinement of B'. Similarly, any bijective f is an isomorphism from the trivial partition to the trivial partition, and thus from a coarsening of any partition to a coarsening of any partition.

We can try to fix this by excluding the singleton and trivial partitions from the definitions — but this just moves us into the lattice a bit. The same basic problem remains: those definitions aren't discriminating enough to be useful.

Prop 2.11: Given (S, B) and (S', B') and morphism (surjective morphism) $f : S \to S'$ from B to B': (i) f is a morphism (surjective morphism) from f to every coarsening of B' and (ii) f is a morphism (surjective morphism) from every refinement of B to B'.

Pf: Suppose f is a morphism from B to B'. (i) Let B'_C be a coarsening of B'. As a morphism to B', each $f(b) \subseteq b'$ for some $b' \in B'$. Since B'_C is a coarsening, $b' \subseteq b'_c$ for some $b'_c \in B'_C$. Therefore $f(b) \subseteq b'_c$, so f is a morphism to B'_C. If f is surjective, it is a surjective morphism. (ii) Let B_R be a refinement of B. As a refinement, each $b_r \subseteq b$ for some $b \in B$. As a morphism from B, each $f(b) \subseteq b'$ for some b'. So $f(b_r) \subseteq f(b) \subseteq b'$, and $f(b_r) \subseteq b'$. So f is a morphism from B_R to B'. If f is surjective, this is a surjective morphism.

Prop 2.12: Given (S, B) and (S', B') and isomorphism $f : S \to S'$ from B to B': (i) for every coarsening B_C of B, f is an isomorphism from B_C to some coarsening of B', (ii) for every refinement B_R of B, f is an isomorphism from B_R to some refinement of B', (iii) for every coarsening B'_C of B', f is an isomorphism from some coarsening of B to B'_C, and (iv) for every refinement B'_R of B', f is an isomorphism from some refinement of B to B'_R.

Pf: We are given that f is an isomorphism, so it is bijective and $g = f'|_B = B'$ is bijective and each $f|_b$ is bijective. (i) Let B_C be a coarsening of B. Then define $B'_C \equiv f'(B_C)$. Since f is bijective, f' is injective, and $f'|_{B_C} = B'_C$ is bijective. We therefore have an isomorphism from B_C to B'_C. Consider $b' \in B'$. Since f is an isomorphism from B to B', $f^{-1}(b') = b$ for some $b \in B$. This $b \subseteq b_c$ for some b_c in B_C since B_C is a coarsening of B. Since f is bijective, $b' = f(b) \subseteq f(b_c) = b'_c$ for some $b'_c \in B'_C$ (by the construction of B'_C). So $b' \subseteq b'_c$ and B'_C is a coarsening of B'. (ii) Let B_R be a coarsening of B. Then define $B'_R \equiv f'(B_R)$. Since f is bijective, f' is injective, and $f'|_{B_R} = B'_R$ is bijective. We therefore have an isomorphism from B_R to B'_R. Consider $b'_r \in B'_R$. Since f is an isomorphism from B_R to B'_R, $f^{-1}(b'_r) = b_r$ for some $b_r \in B_R$. This $b_r \subseteq b$ for some $b \in B$ since B is a coarsening of B_R. However, $f(b) = b'$ for some $b' \in B$ since f is an isomorphism from B to B'. Since f is bijective, $b'_r = f(b_r) \subseteq f(b) = b'$. So $b'_r \subseteq b'$ for some b' and B'_R is a refinement of B'. (iii,iv) Since f is an isomorphism from B to B', its inverse is an isomorphism from B' to B. Just apply (i) and (ii) to f^{-1} to get isomorphisms from B'_C to some B_C or from B'_R to some B_R. The inverses of these (i.e. f) then are isomorphisms from that B_C to B'_C or from that B_R to B'_R.

Prop 2.13: We gain no generality from (b) and (c). They are already embodied in our definitions of uncoarsening/unrefining nonmorphisms and coarsening/refining morphisms.

Pf: (b) Suppose that f is an isomorphism from a refinement B_R of B to a coarsening B'_C of B'. Then by (i), it is an isomorphism from B (a coarsening of B_R) to some coarsening of B'_C. However, any coarsening of B'_C is a coarsening of B' too. f thus is an isomorphism from B to a coarsening of B', making it an uncoarsening/unrefining nonmorphism. (c) Suppose that f is an isomorphism from a coarsening B_C of B to a refinement B'_R of B'. Then by (ii), it is an isomorphism from B (a refinement of B_C) to some refinement of B'_R. However, a refinement of B'_R is also a refinement of B'. f thus is an isomorphism from B to a refinement of B', making it a coarsening/refining morphism.

We therefore see that the apparent generality of (a)-(d) is an illusion. These cases are either vacuous or already covered by our existing definitions. It turns out that our four named definitions aren't independent either. In fact, they reduce to two definitions which can be regarded as inverses of one another. In totem, they embody a single concept, much as refinement and coarsening themselves do. These relationships are established in propositions 2.14 and 2.15 below.

Prop 2.14: (i) f is an uncoarsening nonmorphism iff it is an unrefining nonmorphism, (ii) f is a coarsening morphism iff it is a refining morphism, (iii) if f is a coarsening/refining morphism, it is a bijective (and thus surjective) morphism from B to B', and (iv) if f is a nonisomorphism uncoarsening/unrefining nonmorphism, it is not a morphism from B to B'.

Pf: (i) (forward) If f is an uncoarsening nonmorphism, then f is an isomorphism from B to some B'_C. Proposition 2.12 (part iv) then tells us that f is an isomorphism from some B_R to B', since B' is a refinement of B'_C. This makes it an unrefining nonmorphism. (backward) Same tactic, but using part i of proposition 2.12. (ii) (forward) Same tactic, but using part iii of proposition 2.12. (backward) Same tactic, but using part ii of proposition 2.12. (iii) if f is a coarsening morphism then it is an isomorphism from B to some B'_R, and thus bijective. This makes it a bijective (and thus surjective) morphism from B to B'_R. By proposition 2.11, it also is a surjective morphism from B to every coarsening of B'_R, including to B'. (iv) Let f be an nonisomorphism unrefining nonmorphism. Then it is an isomorphism from some B_R to B'. Since B is a proper coarsening of B_R, some class of B will split across multiple classes of B'.

It trivially follows that f is a nonisomorphism uncoarsening nonmorphism iff it is a nonisomorphism unrefining nonmorphism and f is a nonisomorphism coarsening morphism iff it is a nonisomorphism refining morphism.

Bear in mind that the same f is serving in multiple capacities here. If f is a coarsening morphism from B to B', then it is an isomorphism from B to some B'_C and a surjective morphism from B to B' and an isomorphism from some B_R to B'. Although f is the same, with the same information content, the induced g varies. The g from B to B'_C has the lowest information content, that from B to B' has more information, and the g from B_R to B' has the most information.

In light of proposition 2.14, we'll just speak of "coarsening morphisms" and "uncoarsening nonmorphisms". As the following propositions make clear, the coarsening morphisms from B to B' are precisely the bijective morphisms between B and B', and the uncoarsening nonmorphisms are their inverses.

Prop 2.15: f is a coarsening morphism from B to B' iff f^{-1} is an uncoarsening nonmorphism from B' to B.

Since f is bijective for coarsening morphisms and uncoarsening nonmorphisms, $f^{-1} : S' \to S$ exists.

Pf: Suppose f is a coarsening morphism from B to B'. Then it is an isomorphism from B to some B'_R that refines B'. This means that f^{-1} is an isomorphism from B'_R to B. I.e., it is an unrefining nonmorphism from B' to B, which we saw is the same as an uncoarsening nonmorphism. Now, suppose f is an uncoarsening nonmorphism from B to B'. Then it is an isomorphism from B to some B'_C that coarsens B'. Again, f^{-1} is an isomorphism, now from B'_C to B. It therefore is a refining morphism from B' to B, which we saw is the same as a coarsening morphism.

It follows trivially that f is a nonisomorphism coarsening morphism from B to B' iff it is a nonisomorphism uncoarsening nonmorphism from B' to B.

Prop 2.16: Given (S, B) and (S', B'), f is a coarsening morphism from B to B' iff it is a bijective morphism from B to B'.

Pf: (forward) Suppose f is a coarsening morphism. Since f is an isomorphism between B and some B'_R, it must be bijective. We know from proposition 2.14, part iii, that it is a surjective morphism from B to B'. It therefore is a bijective morphism from B to B'. (backward) Suppose f is a bijective morphism from B to B'. Define $B'_R \equiv \{f(b); b \in B\}$. Since f is a surjective morphism to B', each $f(b) \subseteq b'$ for some $b' \in B'$. Since f is bijective, $f(b_1) \cap f(b_2) = \emptyset$ for $b_1 \neq b_2$. I.e., each b' has a disjoint cover by elements of B'_R. So B'_R is a refinement of B'. The g induced to B'_R is — by construction — $f'|_B$ and is bijective. So f is an isomorphism from B to B'_R, and thus a coarsening morphism from B to B'.

Prop 2.17: Any bijective $f : S \to S'$ is (i) a coarsening morphism from the singleton partition on S to every partition of S', (ii) a coarsening morphism from every partition of S to the trivial partition of S', (iii) an uncoarsening nonmorphism from the trivial partition of S to every partition of S', and (iv) an uncoarsening nonmorphism from every partition of S to the singleton partition of S'.

In particular, Id_S is a coarsening morphism from the singleton partition of S to every partition of S and from every partition of S to the trivial partition of S.

Pf: (i) From proposition 2.6, we know that a bijective f is an isomorphism from any B to its unique partner B'. It therefore takes the singleton partition on S to some partner B'. However, that partner can only be the singleton partition on S', because every singlet has to map to a singlet. Since every partition of S' is a coarsening of the singleton partition, f is a coarsening morphism from the singleton partition on S to every partition on S'. (ii) Similarly, it is an isomorphism between the trivial partitions, and every partition is a refinement of the trivial partition, so f is a coarsening morphism from every partition of S to the trivial partition of S'. (iii) and (iv) follow if we apply (i) and (ii) to f^{-1} and then use proposition 2.15.

In fact, for any B on S, a bijective f establishes a bijection between the set of coarse-grainings (on S) of B and the set of coarse-grainings (on S') of its isomorphic partner $f'(B)$.

This tells us that a bijection cannot serve as an ordinary (i.e. not coarsening) morphism between *any* partitions. Given B and B', such an f is either an isomorphism, a nonisomorphism coarsening morphism in one or the other direction (and thus a nonisomorphism uncoarsening nonmorphism in the opposite direction), or not a morphism in either direction.

One consequence of all this is that no nonisomorphism morphism can have a meaningful inverse morphism. To even speak of an inverse, we need f to be bijective. This means that f must be a coarsening morphism if it is a morphism. However, the inverse of a nonisomorphism coarsening morphism is a nonisomorphism uncoarsening nonmorphism, which can never be a morphism. This is, of course, no surprise in light of proposition 2.4.

Coarsening morphisms compose to coarsening morphisms, and uncoarsening nonmorphisms compose to uncoarsening nonmorphisms.

Prop 2.18: Consider B_1 on S_1, B_2 on S_2, and B_3 on S_3. (i) The composition of coarsening morphisms is a coarsening morphism. If either is not an isomorphism, then the result is not an isomorphism. (ii) The composition of uncoarsening nonmorphisms is an uncoarsening nonmorphism. If either is not an isomorphism, then the result is not an isomorphism.

Pf: (i) Let $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ be coarsening morphisms from B_1 to B_2 and B_2 to B_3. As a composition of bijections, $f_2 \circ f_1 : S_1 \to S_3$ is bijective. By definition, f_1 and f_2 are surjective morphisms. By proposition 1.5, surjective morphisms (aka maps 'into') compose to a surjective morphism. So, $f_2 \circ f_1$ is a bijective morphism, and thus a coarsening morphism. If f_1 and f_2 are isomorphisms, then we know that the induced maps g_1 and g_2 are bijective and thus compose to a bijective induced map. We thus have composition of isomorphisms. We saw in proposition 2.9 that if either is not an isomorphism, the composition isn't.

Pf: (ii) If f_1 and f_2 are uncoarsening nonmorphisms, then consider $f_2 \circ f_1$. The inverse of an uncoarsening nonmorphism is a coarsening morphism, so $(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$ is a composition of coarsening morphisms. Its inverse, $f_2 \circ f_1$, therefore is an uncoarsening nonmorphism. Suppose at least one of them is not an isomorphism. Then its inverse isn't an isomorphism, and part (i) tells us that we cannot compose the coarsening morphisms to an isomorphism. The inverse of that composition therefore isn't an isomorphism either.

Note that the composition of a coarsening morphism with an uncoarsening nonmorphism (in either order) need not be a morphism or even an uncoarsening nonmorphism.

Ex. (isomorphism): Let $S = S'$, let $B = B'$, and let f be a coarsening morphism. f^{-1} is an uncoarsening nonmorphism, and $f \circ f^{-1} = Id_S$ is an isomorphism from B to B'.

Ex. (non-morphism): Let $S = S' = \{1, 2, 3, 4, 5, 6\}$, let $B = \{(1, 2, 3), (4, 5, 6)\}$ and $B' = \{(1, 2, 3, 4, 5, 6)\}$ and $B'' = \{(1, 2), (3, 4), (5, 6)\}$. Then Id_S is a coarsening morphism from B to B' and an uncoarsening nonmorphism from B' to B''. $Id_S \circ Id_S = Id_S$ and $Id_S^{-1} = Id_S$, so all the relevant maps are just Id_S. However, Id_S is not a morphism from B to B'' or from B'' to B. Nor is it an uncoarsening nonmorphism in either direction.

Finally, we observe the following about the induced maps g and $\{h_b\}$ for an isomorphism vs a nonisomorphism coarsening morphism vs a nonisomorphism uncoarsening nonmorphism.

Prop 2.19: (i) $f : S \to S'$ is an isomorphism from B to B' iff f is a bijective morphism from B to B' and g is injective. (ii) In that case, g and all the h_b's are bijective and f^{-1} is an isomorphism from B' to B, with corresponding induced maps g^{-1} and $\{h_b^{-1}\}$. (iii) f is a nonisomorphism coarsening morphism from B to B' iff f is a bijective morphism from B to B' and g is not injective. (iv) In that case, g is surjective and each h_b is injective, and f^{-1} is a nonisomorphism uncoarsening nonmorphism from B' to B and has no corresponding induced map (since it's not a morphism).

Note that when we speak of induced maps for the coarsening morphism and uncoarsening nonmorphism, we mean between B and B' (or B' and B, depending which it applies to), *not* between the relevant refinements/coarsenings. Since such an f is an isomorphism between those relevant refinements or coarsenings and either B or B', obviously the induced g for *that* pair of partitions is always bijective, as are all the h_b's.

Pf: (i,iii) Proposition 2.16 tells us that f is a coarsening morphism iff it is a bijective morphism. From def 12, f is an isomorphism from B to B' iff it is a bijective morphism and g is injective (and thus bijective).

Pf: (ii) If f is a morphism, g is surjective. If f is an isomorphism, f and g are bijective. If f is bijective, its restriction to any $h_b = f|_b$ is bijective to its image $f(b)$. Since g is bijective, $f(b) = g(b)$, so the image is a whole class of B'. For an isomorphism, $g = f'|_B$. We saw in proposition 1.1 that $f'^{-1} = (f^{-1})'$ overall, so they're equal when restricted to B'. It follows that the map induced by f^{-1} is g^{-1}. Since $h_b : b \to g(b)$ is bijective relative to $g(b)$ and equal to $f|_b$, the map $h_b^{-1} : b' \to g^{-1}(b')$ is bijective relative to b and equal to $f^{-1}|_{b'}$.

Pf: (iv) Since f is bijective, proposition 2.1 tells us that g is surjective and the h_b's are injective. The rest just restates proposition 2.15.

2.7. **Table and Graph of Definitions.** Let's summarize the various definitions of maps and their relationships. The following table and graph illustrate these relationships. We also provide a set of examples.

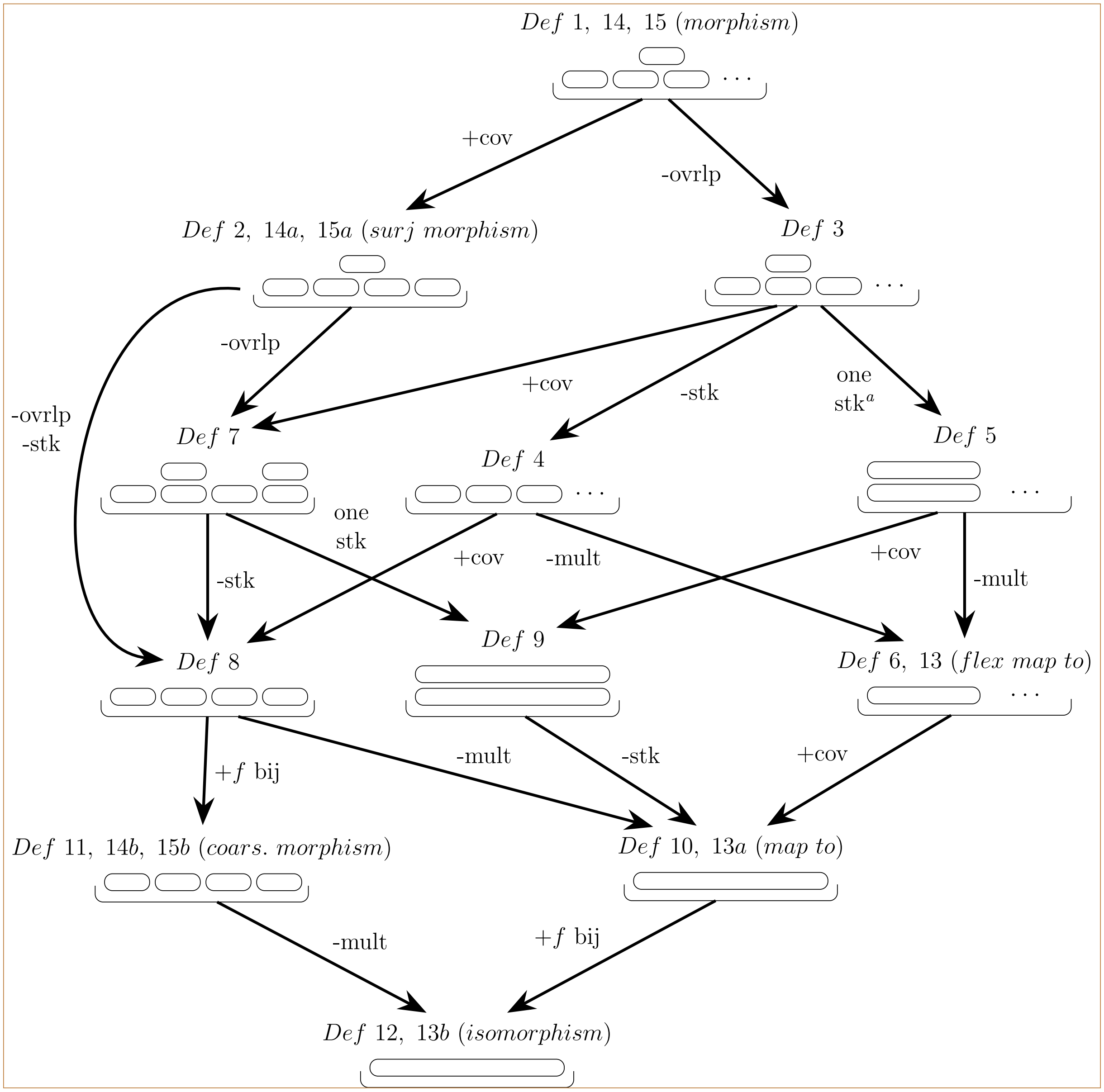
Table 1 is a list of our definitions. Note that we only list the top type as the 'name'. If def B is derived from def A, and def A is a 'foo' (in the 'name' column), then so is def B — but we won't say so explicitly unless it has a special name of its own. Also note that nonisomorphism uncoarsening nonmorphisms don't appear in this table because they are not morphisms between B and B', and everything in this table is a morphism of some sort.

TABLE 1. Comparison of Morphism Definition Candidates

	Functions				$f(b)$ tiles[2]. per b'				Other			
Group	f	g	h_b	$g \stackrel{?}{=} f'\|_B$	cov	ovrlp	stk	mult	tightens	top	meas	name(s)
1, 14, 15	—	surj	—	no	no	yes	yes	yes	—	cont	meas	- morph - flexible map into
2, 14a, 15a	surj	surj	—	no	yes	yes	yes	yes	1, 14, 15	surj cont	surj meas	- surj morph - map into
3	—	surj	—	no	no	no	yes	yes	1, 14, 15	—	—	—
4	—	surj	—	no	no	no	no	yes	3	—	—	—
5	—	surj	—	no	no	no	yes	yes	3	—	—	—
6, 13	—	bij	—	no	no	no	no	no	4 or 5	—	—	- flexible map to
7	surj	surj	—	no	yes	no	yes	yes	2 or 3	—	—	—
8	surj	surj	—	no	yes	no	no	yes	2, 14a, 15a or 4 or 7	—	—	—
9	surj	surj	surj	yes	yes	no	yes	yes	5 or 7	—	—	—
10, 13a	surj	bij	surj	yes	yes	no	no	no	6, 13 or 8 or 9	—	—	- map to
11, 14b, 15b	bij	surj	inj	no	yes	no	no	yes	8	—	—	- coarsening morphism
12, 13b	bij	bij	bij	yes	yes	no	no	no	10 or 11	homeo	bimeas	- isomorph

'cov= must the tiles cover b', 'ovrlp'= can tiles overlap in part, 'stk'= can tiles overlap in whole, and 'mult'=do we allow more than one tile per b'. Note that 'ovrlp' asks whether we specifically allow partial overlaps, not just any form of overlap. This differs from question (iii) of the four questions we asked earlier. There, we asked whether *any* overlaps (partial or total) were allowed. Here, we're just asking whether partial overlaps are allowed. I.e., neither 'stk' nor 'ovrlp' implies the other. The information content of the earlier questions (ii)+(iii) is the same as that of 'ovlp'+'stk' here, but it is framed differently. Just like in our earlier discussion, our four flags are neither independent nor complete. They represent 8 possibilities in total, 4 for each of the two 'cov' values — but we additionally split one of those 4 cases ('ovrlp'=no, 'stk'=yes, 'mult'=yes) into a case with a single stack and a case with > 1 stacks. This gives us 5 possibilities per value of 'cov'. We also have 2 additional defs where we require a bijective f. These have 'cov'=yes, 'ovrlp'=no, 'stk'=no, 'mult'=no.

Figure 1. Relationship Between Morphism Definition Candidates



[a]By this, we mean that we allow only a single stack.

The following examples illustrate cases which fit each definition but nothing stricter. We're always assuming that g is surjective, and each definition specifies constraints on the way $f(b)$'s fill a given b'.

Ex. Def 1, 14, 15: Let $S = S' = \{1,2,3,4,5,6\}$ and $B = \{(1,2),(3,4),(5,6)\}$ and $B' = \{(1,2,3,4,5,6)\}$ and let $f : (1,2,3,4,5,6) \to (1,2,2,3,6,6)$.

Ex. Def 2, 14a, 15a: Let $S = \{1,2,3,4,5,6\}$ and $S' = \{1,2,3,4,5\}$ and $B = \{(1,2),(3,4),(5,6)\}$ and $B' = \{(1,2,3,4,5)\}$ and let $f : (1,2,3,4,5,6) \to (1,2,2,3,4,5)$.

Ex. Def 3: Let $S = S' = \{1,2,3,4,5,6\}$ and $B = \{(1,2),(3),(4),(5,6)\}$ and $B' = \{(1,2,3,4,5,6)\}$ and let $f : (1,2,3,4,5,6) \to (1,1,1,2,3,4)$.

Ex. Def 4: Let $S = S' = \{1,2,3,4,5,6\}$ and $B = \{(1,2),(3),(4),(5,6)\}$ and $B' = \{(1,2,3,4,5,6)\}$ and let $f : (1,2,3,4,5,6) \to (1,1,2,3,4,5)$.

Ex. Def 5: Let $S = S' = \{1, 2, 3, 4, 5, 6\}$ and $B = \{(1, 2, 3), (4, 5, 6)\}$ and $B' = \{(1, 2, 3, 4, 5, 6)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 3, 1, 2, 3)$.

Ex. Def 6, 13: Let $S = S' = \{1, 2, 3, 4, 5, 6\}$ and $B = \{(1, 2, 3, 4, 5, 6)\}$ and $B' = \{(1, 2, 3, 4, 5, 6)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 3, 1, 2, 3)$.

Ex. Def 7: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S' = \{1, 2, 3, 4\}$ and $B = \{(1), (2), (3), (4), (5), (6)\}$ and $B' = \{(1, 2, 3, 4)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 1, 2, 3, 3, 4)$.

Ex. Def 8: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S' = \{(1, 2, 3, 4)\}$ and $B = \{(1, 2), (3, 4), (5, 6)\}$ and $B' = \{(1, 2, 3, 4)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 3, 3, 4, 4)$.

Ex. Def 9: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S' = \{1, 2\}$ and $B = \{(1, 2), (3, 4), (5, 6)\}$ and $B' = \{(1, 2)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 1, 2, 1, 2)$.

Ex. Def 10, 13a: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S' = \{1, 2, 3\}$ and $B = \{(1, 2, 3, 4, 5, 6\}$ and $B' = \{(1, 2, 3)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 3, 1, 2, 3)$.

Ex. Def 11, 14b, 15b: Let $S = S' = \{1, 2, 3, 4, 5, 6\}$ and $B = \{(1, 2), (3, 4), (5, 6)\}$ and $B' = \{(1, 2, 3, 4, 5, 6)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 3, 4, 5, 6)$.

Ex. Def 12, 13b: Let $S = S' = \{1, 2, 3, 4, 5, 6\}$ and $B = \{(1, 2, 3, 4, 5, 6)\}$ and $B' = \{(1, 2, 3, 4, 5, 6)\}$ and let $f : (1, 2, 3, 4, 5, 6) \rightarrow (6, 5, 4, 3, 2, 1)$.

2.8. **Properties of Isomorphisms and Morphisms.**

Now that we've established our notions of morphism, surjective morphism, and isomorphism, let's consider some properties of them.

We saw that 'maps into' is the same as 'surjective morphism', but 'maps to' is not the same as 'isomorphism'. The difference between the latter two is bijectivity of f, as embodied in the following proposition.

Prop 2.20: f is an isomorphism of B and B' iff f maps B 'to' B' and is injective.

This follows immediately from the definitions.

We already saw that 'maps to' and 'maps into' compose. The following proposition incorporates and extends this. The gist is that maps of type FOO compose to a map of type FOO, where FOO is any of the named types.

Prop 2.21: (i) The composition of morphisms (aka flexible maps 'into') is a morphism, (ii) the composition of surjective morphisms (aka maps 'into') is a surjective morphism, (iii) the composition of isomorphisms is an isomorphism, (iv) the composition of coarsening morphisms is a coarsening morphism, (v) the composition of uncoarsening nonmorphisms is an uncoarsening nonmorphism, (vi) the composition of maps 'to' is a map 'to', and (vii) the composition of flexible maps 'to' is a flexible map 'to'. (viii) The (left or right) composition of an isomorphism with a map of any of these types is a map of that same type. (ix) the composition of nonisomorphism coarsening morphisms is a nonisomorphism coarsening morphism. (x) the composition of nonisomorphism uncoarsening nonmorphisms is a nonisomorphism uncoarsening nonmorphisms.

I.e. all the named types are closed under composition. Bear in mind that none of these are groups in general because they are maps between different S's.

Pf: (ii) and (vi) just restate proposition 1.5. (i) and (vii) restate the addendum to that proposition (stated in the same section) for non-surjective f's. (iv) and (v) just restate proposition 2.18. (iii) just restates proposition 2.9. (ix) and (x) follow trivially from (iii), (iv), (v), and proposition 2.18.

Pf: (viii) An isomorphism is a morphism, a surjective morphism, a map 'to', and a flexible map 'to'. As such, it composes under the relevant rule. If it composes with a nonisomorphism coarsening morphism, it gives a nonisomorphism coarsening morphism via proposition 2.18. Let f_2 be an isomorphism, which means f_2^{-1} is as well. If f_1 is an uncoarsening nonmorphism, then f_1^{-1} is a coarsening morphism, so $f_2^{-1} \circ f_1^{-1}$ is a coarsening morphism, which makes $f_1 \circ f_2$ an uncoarsening nonmorphism. The same works on the other side by using $f_1^{-1} \circ f_2^{-1}$.

2.8.1. *Classification of partitions modulo isomorphism.* From what we've discussed, it is clear that, for all practical purposes, only the cardinalities matter when it comes to partitions. Modulo isomorphism, we only care about $|S|$, $|B|$, and the cardinalities of the classes of $|B|$. More precisely, given the following information, we have a unique (modulo isomorphism) specification of a partition:

- $|S|$
- $|B| \leq |S|$
- For each $x \leq |S|$, the number n_x of classes of cardinality x (and subject to the constraint that $\sum_x x \cdot n_x = |S|$ using cardinal arithmetic).

The following proposition formalizes this:

Prop 2.22: (S, B) and (S', B') are isomorphic iff $|S| = |S'|$, $|B| = |B'|$, and there exists a bijection k between B and B' s.t. $|k(b)| = |b|$ for all $b \in B$.

Pf: We'll assume the axiom of choice in this proof. (forward) Let $f : S \to S'$ be an isomorphism between B and B'. Then f is bijective, so $|S| = |S'|$, and $g = f'|_B$ is bijective to B', so $|B| = |B'|$. Moreover, $f|_b$ is bijective to its image $g(b) = f'(b)$. $k \equiv f'|_B$ therefore is the relevant bijection. (backward) Let $|S| = |S'|$ and $|B| = |B'|$ and let a suitable bijection k exist s.t. $|k(b)| = |b|$ for all $b \in B$. Since B is a partition, we may construct $f|_b$ piece by piece and then glue them together. Since $|k(b)| = |b|$, pick a bijection (via the axiom of choice) $h_b : b \to k(b)$ and define $f|_b = h_b$. We now have a bijection $f : S \to S'$. By construction, $f'|_B = k$, and $f|_b$ is bijective to the induced $g(b) = f'(b)$. We therefore have an isomorphism.

2.8.2. *The partitions from which f is a surjective morphism to B'.* Consider a given surjective $f : S \to S'$ and a partition B' of S'. f maps some set of partitions of S 'into' B' and possibly maps some partition of S 'to' B'. We know from proposition 1.2 that the former set is nonempty (and contains at least the singleton partition on S) and that the latter (if it exists) is unique. We also know (from proposition 2.20) that if f is bijective and maps 'to' B' from some partition B, it is an isomorphism between B and B'. Let's now examine more closely the set of partitions of S that are mapped by f 'into' B'.

We observed in proposition 1.7 that the pullback f^*B' is a partition of S, and in proposition 1.8 that if f is surjective then f maps f^*B' 'to' B', and every B that f maps 'into' B' is a refinement of f^*B'. Let's now reframe this in our current language.

Prop 2.23: Let $f : S \to S'$ be surjective and let B' be a partition of S'. Then (i) $f^*B' = \{f^{-1}(b'); b' \in B'\}$ is a partition of S, (ii) f maps f^*B' (and no other partition) 'to' B', (iii) f is an isomorphism between f^*B' and B' iff f is bijective, and (iv) f is a (necessarily surjective) morphism from B to B' iff B is a refinement (proper or not) of f^*B', and (v) if f is bijective, then f is a nonisomorphism morphism from

B to B' iff it is a nonisomorphism coarsening morphism iff B is a proper refinement of f^*B'.

Pf: (i) rephrases proposition 1.7, (ii) rephrases proposition 1.8 part i, (iii) follows from proposition 2.20, and (iv) follows from proposition 1.8 part ii and proposition 1.3 part i. We obtain (v) as follows: if f is bijective then by (ii) it is an isomorphism from f^*B' to B'. It therefore is an isomorphism from a coarsening of B to B' and (by definition) a refining morphism, which proposition 2.14 tells us is the same as a coarsening morphism. From (iv), f is a morphism from B to B' iff B is a refinement of f^*B', so morphism = coarsening morphism = B is refinement in this case. The nonisomorphism part propagates as well, since nonisomorphism = nonisomorphism coarsening morphism = B is a *proper* refinement.

The situation therefore is simple:

- B' has a unique maximally-coarse partner f^*B' which f maps 'to' it.
- If f is bijective, then this is the (unique) isomorphic partner to B' under f.
- f is a morphism from B to B' iff B is a refinement (proper or not) of f^*B'.

This defines a map $Pull_f : Par(S') \to Par(S)$, which takes each B' to its unique maximally-coarse partner f^*B' under f.

2.8.3. *The partitions to which f is a surjective morphism from B.* Let's conduct a similar exercise with a surjective $f : S \to S'$ and a partition B of S. What partitions does f map B 'into' or 'to'? Is there a unique partner?

We know that $f'(B)$ need not be a partition of S' because the $f(b)$'s can overlap. Therefore, we don't have the obvious counterpart of the pullback f^*B'. We nonetheless have a means to construct a unique partner to B under f, using the methods developed in section 1.12.

Prop 2.24: Let $f : S \to S'$ be surjective and let B be a partition of S and let $B'_G \equiv G(f'(B))$ (our meet-like operation from section 1.12). Then (i) B'_G is a partition of S', (ii) f is a morphism from B to B' iff B' is a coarsening (proper or not) of B'_G, (iii) if f is bijective then $G(f'(B)) = f'(B)$ and is the isomorphic partner of B, (iv) if f is bijective, then f is a nonisomorphism morphism from B to B' iff it is a nonisomorphism coarsening morphism iff B' is a proper coarsening of B'_G, and (v) if f maps B 'to' B' then $B' = B'_G$.

Pf: (i) and (ii) just restate proposition 1.18. (iii) If f is bijective, then $f'(B)$ is a partition of S', and $G(f'(B)) = f'(B)$ automatically. In that case, $f'(B)$ is the unique isomorphic partner of B' because the induced g is bijective. (iv) We know from (iii) that f is an isomorphism from B to B'_G. From (ii) we know that it is a morphism from B to B' iff B' is a coarsening of B'_G. This means that f is an isomorphism from B to a refinement of B'. By definition, this is a coarsening morphism. By (ii) morphism = coarsening morphism = coarsening of B'_G. And by (iii) and our earlier results, nonisomorphism morphism = nonisomorphism coarsening morphism = proper coarsening of B'_G. (v) Let f map B 'to' B'. A map 'to' is a map 'into' as well and thus a surjective morphism. By (ii), this means that B' is a coarsening (proper or not) of B'_G. However, we also know from proposition 1.3 part iii that f is a surjective morphism (aka 'map into') from B to every coarsening of B'. This means that $B' = B'_G$. Otherwise, there would be some surjective morphism to a non-coarsening of B' or to a non-coarsening of B'_G.

If f is non-injective, then it can't be an isomorphism to any partition. However, it still may (or may not) be a map 'to' some B'. This depends on the form of the non-injectivity. If f is confined to intra-class noninjectivity (i.e. the induced g is $f'|_B$ and is bijective, but f can be noninjective within each $f(b)$), then f maps B 'to' $f'(B)$. In that case, the relevant B' equals B'_G.

In this direction, things don't look as nice. We are not guaranteed that f 'maps to' any B'. We nonetheless can identify a unique maximally-refined partner B'_G.

- B has a unique maximally-refined partner $B'_G \equiv G(f'(B))$.

- If f is bijective, then B'_G is the (unique) isomorphic partner to B under f.
- f is a morphism from B to B' iff B' is a coarsening (proper or not) of B'_G.
- A non-injective f may or may not map B 'to' some B'. If it does, then $B' = B'_G$.

This defines a map $Push_f : Par(S) \to Par(S')$, which takes each B to its maximally-refined partner $G(f'(B))$ under f.

2.8.4. *Roundtrips.* For a given f, we have maps $Pull_f : Par(S') \to Par(S)$ and $Push_f : Par(S) \to Par(S')$. The natural question to ask is whether they are inverses. The answer is 'not quite'. The following proposition addresses how the roundtrips behave.

Prop 2.25: Let $f : S \to S'$ be surjective: (i) $Push_f \circ Pull_f = Id_{Par(S')}$ and (ii) $Pull_f \circ Push_f$ takes B to a coarsening (proper or not) of B.

Pf: (i) By construction, f^*B' is a partition of S that is bijective with B'. Therefore, $f'(f^*B') = B'$. Since B' is a partition, $G(B') = B'$. (ii) $G(f'(B))$ is a partition of S' s.t. each $f(b) \subseteq b'$ for some b'. Therefore $f^{-1}(b') \supseteq b$. I.e., b is a subset of some class of $(Pull_f \circ Push_f)(B)$. This means that B is a refinement of $(Pull_f \circ Push_f)(B)$.

Ex. of (ii): Let $S = \{(1,2,3,4)\}$, $S' = \{1,2\}$, $B = \{(1),(2),(3),(4)\}$, and $f : (1,2,3,4) \to (1,1,2,2)$. Then $Push_f(B) = \{(1),(2)\}$ and $(Pull_f \circ Push_f)(B) = \{(1,2),(3,4)\}$, which is a coarsening of B.

Let's now consider some properties of $Pull_f$ and $Push_f$.

Prop 2.26: $Pull_f$ is injective. It is bijective iff f is bijective.

Note that if f is merely surjective, $Pull_f$ need not be surjective. Consider a given B. If some of the $f(b)$'s overlap (in whole or in part), then there can be no B' with $f^*B' = B$. We know that f^*B' maps 'to' B', which means that the induced g must be bijective. However, it is impossible for B to map 'to' something if its classes overlap (thus making g noninjective).

Pf: Suppose B'_1 and B'_2 have $f^*B'_1 = f^*B'_2$. Call this B. Then B maps 'to' both B'_1 and B'_2. However, by proposition 1.2 we know that B can map 'to' at most one partition under f. Therefore, $Pull_f$ is injective. Now, suppose that f is bijective. Then $f'(B)$ is bijective with B and is the unique isomorphic partner to B and vice versa. This establishes $Pull_f$ as a bijection between $Par(S')$ and $Par(S)$. Conversely, suppose that $Pull_f$ constitutes a bijection between $Par(S')$ and $Par(S)$. Suppose that f is surjective but not injective. Let $f(x_1) = f(x_2)$ for $x_1 \neq x_2$. Let $B'_{S'}$ be the singleton partition of S' and B_S be the singleton partition of S. As a bijection, $Pull_f$ maps every B' to some B, so let $B_1 \equiv Pull_f(B'_{S'})$. If $B_1 = B_S$ then we are done, because we have a bijection between singletons which exactly corresponds to f^{-1}. Suppose that $B_1 \neq B_S$. Then B_1 is a coarsening of B_S. Because $Pull_f$ is a bijection, we know that some B' must pull-back to B_S. However, that is impossible. Since every B' is a coarsening of $B'_{S'}$, every f^*B' is a coarsening of $f^*B'_{S'}$. This means that no B' can pull back to B_S, contradicting our premise. Therefore, $B_{S'}$ pulls back to B_S, and f is a bijection.

Prop 2.27: $Push_f$ is surjective. It is bijective iff f is bijective.

If f is non-injective, then we can (but need not) have $f(b_1) \cap f(b_2) \neq \emptyset$ for some $b_1 \neq b_2$. Suppose this is the case. Let B_1 be the partition of S obtained from B by merging b_1 and b_2. Then f would take B and B_1 to the same B', and $Push_f$ would be noninjective.

Pf: Consider B' on S'. We know from proposition 2.25 that $(Push_f \circ Pull_f) = Id_{Par(S')}$. So $Push_f$ takes f^*B' to B'. Therefore, $Push_f$ is surjective. Suppose that f is bijective. In that case, f is an isomorphism between B and $f'(B)$. Since the isomorphic partner is unique on both ends, $Push_f$ pairs up elements of $Par(S)$ and $Par(S')$ as a bijection. Conversely, suppose that $Push_f$ is a bijection. If f is noninjective, then $f(x_1) \neq f(x_2)$ for some $x_1 \neq x_2$. Let B_S be the singleton partition on S and let B_1 be the partition obtained from B_S by merging (x_1) and (x_2) into a single class (x_1, x_2). Since $f'(B_S) = f'(B_1)$ as a set, $G(f'(B_S)) = G(f'(B_1))$. This means that $Push_f(B_S) = Push_f(B_1)$, violating our premise that $Push_f$ is bijective.

Prop 2.28: The following 3 conditions are equivalent: (i) f is a bijection, (ii) $Push_f$ is a bijection between $Par(S)$ and $Par(S')$, and (iii) $Pull_f$ is a bijection between $Par(S')$ and $Par(S)$. If any (and thus all) hold, then (iv) $Push_f = Pull_f^{-1} = Pull_{f^{-1}}$ and (v) $Pull_f = Push_f^{-1} = Push_{f^{-1}}$.

Pf: (i)↔(ii) is just the content of proposition 2.27. (i)↔(iii) is just the content of proposition 2.26. (i)↔(iii) follows automatically by the transitivity of iff. (iv) Suppose f is bijective. Then $f'(B) = B'$ is the isomorphic partner to B and $f^*(f'(B)) = B$, so $Push_f = Pull_f^{-1}$ fully (not just one-sided). Consider $Pull_{f^{-1}}(B) = (f^{-1})^*(B)$. Since $Pull_f \circ Pull_{f^{-1}}(B) = f^*((f^{-1})^*(B))$ takes $b \to (f^{-1})^{-1}(b) = f(b) \to f^{-1}(f(b)) = b$, we have $Pull_f \circ Pull_{f^{-1}} = Id_{Par(S)}$, and it is easy to see that the same holds on the other side. I.e., $Pull_{f^{-1}} = Pull_f^{-1}$. (v) We've already shown that $Pull_f = Push_f^{-1}$. This also means that $Push_{f^{-1}} = Pull_{f^{-1}}^{-1}$. However, we saw that $Pull_{f^{-1}} = Pull_f^{-1}$, so $Pull_{f^{-1}}^{-1} = Pull_f$, and we have $Push_{f^{-1}} = Pull_f$.

2.8.5. *Maximally-refined and Maximally-coarse partners.* Let's reiterate and emphasize an important point from this discussion. $Push_f$ and $Pull_f$ generalize the notion of isomorphic partner from the bijective f case to the surjective f case. This comes at the cost of a loss of symmetry in the definition, since f^{-1} needn't exist. That is why the definitions of $Pull_f(B')$ and $Push_f(B)$ are quite different.

However, this does hint that the notion of $G(f'(B))$ is dual to the pullback f^*B'.

For a surjective (but possibly noninjective) f, we saw that $Pull_f$ is injective but not necessarily surjective. We are guaranteed that no two choices of B' have the same $Pull_f(B')$, but not every B need be the pull-back of some B'. On the other hand, $Push_f$ is surjective but not necessarily injective. We are guaranteed that every B' is $Push_f(B)$ for some B, but it is quite possible that B_1 and B_2 have $Push_f(B_1) = Push_f(B_2)$.

What happens if f is not even surjective? In that case, we saw that f^*B' still is a partition, but f may not be a morphism from it to B'. The problem is that the induced $g : f^*B' \to B'$ need not be surjective.

We can try to require that f is only nonsurjective in a way that keeps g surjective (i.e. satisfies $f'(f^*B') = B'$), but such a constraint is specific to that particular B'. For a nonsurjective f, we can always find *some* partition B' for which $f'(f^*B') \neq B'$ and thus isn't a partition. To accomplish this, group $(S' - \text{Im } f)$ into a single class b', and let B' be any partition with b' as a class. Then $f'(f^*B')$ is nonsurjective to B' since it doesn't take any class of f^*B' to b'. Therefore, the resulting g is non-surjective.

Moreover, $Pull_f$ becomes noninjective if f is nonsurjective. Any partitions that differ only in unmapped-to classes (i.e. classes that are subsets of $(S' - \text{Im } f)$ will have the same f^*B'.

Worse still, if f is not surjective, $Push_f$ no longer maps partitions to partitions.

$G(f'(B))$ need not produce a partition, because $f'(B)$ isn't a cover of S'. Nor is their even a unique B' that contains the incomplete partition $f'(B)$ as a subset. We can split the unmapped-to space (i.e. $(S' - \text{Im } f)$) into classes however we wish. Therefore, $Push_f$ cannot be regarded as a function to $Par(S')$. We may either redefine it as a many-valued map from $Par(S)$ to $Par(S')$, by taking each B to all partitions of S' which contain $f'(B)$ as a subset, or we may view it as a function from $Par(S)$ to $2^{S'}$ (or to the set of sets of disjoint subsets of S, without the covering requirement that defines a partition). Neither of these approaches are useful.

For these reasons, we cannot extend our $Push_f$ and $Pull_f$ maps to non-surjective f's.

This is one major argument in favor of confining ourselves to surjective f's. For a given B', particular non-surjective f's may be palatable (if they induce surjective g's). However, for any given nonsurjective f, there are always partitions B' against which it is problematic.

2.8.6. *Isomorphism Theorems.*

In light of proposition 2.4, we may be tempted to hope for a counterpart for morphisms of the Cantor-Schroeder-Bernstein theorem from set theory. I.e., that if there exist surjective morphisms from B to B' and from B' to B, then there must exist an isomorphism between B and B'. Unfortunately, this is not the case — as the following example illustrates.

Recall the the Cantor-Schroeder-Bernstein thm tells us that if we have injective maps $S \to S'$ and $S' \to S$, there must exist a bijective map between S and S'. This can be proven without the axiom of choice (though most common proofs employ it). There is a corollary that if we are given a pair of opposing surjective functions then we must have a bijection as well. This requires the axiom of choice. Basically, we construct an injective function from the (multi-valued) inverse in each direction and then plug this pair (in the opposite direction from the surjective maps) into the CSB theorem. This is more closely analogous to our case, since we have a pair of surjective morphisms. In fact, the CSB theorem itself tells us that there exists a bijection between S and S'. However, this doesn't mean that there exists an isomorphism between our particular B and B'.

Ex. suppose $S = S' = N$, $B = \{(1,2),(3,4),\dots\}$ and B' is the singleton partition. Clearly, $|S| = |S'|$ and $|B| = |B'|$. Define $f_1(n) = [(n+1)/2]$ (i.e. $f_1(1) = f_1(2) = 1$, $f_1(3) = f_1(4) = 2$, etc) and $f_2 = Id_N$. Then f_1 is a surjective morphism (and, in fact, a map 'to') from (S, B) to (S', B') and f_2 is a surjective morphism (and, in fact, a coarsening morphism) from (S', B') to (S, B). f_1 is surjective and noninjective but induces a bijective g_1, while f_2 is bijective but induces a noninjective g_2. However, there can be no isomorphism between B and B', because no class of B has the same cardinality as any class of B'.

Failing this, we can entertain a weaker alternative (i.e. where we tighten the requirements). For example, we could require an opposing pair of coarsening morphisms. Unfortunately, this doesn't work either — or perhaps we could say it works too well. We cannot have such a pair unless they are both isomorphisms to begin with.

Prop 2.29: Let f_1 be a coarsening morphism from B to B' and f_2 be a coarsening morphism from B' to B. Then f_1 and f_2 are isomorphisms.

Note that f_1 and f_2 may be unrelated. We are not claiming that $f_1 = f_2^{-1}$, though this certainly is a possibility.

Pf: Viewed as a refining morphism, f_1 is an isomorphism from a coarsening B_C of B to B' and f_2 is an isomorphism from B' to some refinement B_R of B. However, isomorphisms compose, so isomorphicness is a transitive relation amongst partitions. This means that, B_R and B_C are isomorphic, which can only happen if both equal B. Therefore, f_1 is an isomorphism from B to B'. Similarly, f_2 is an isomorphism from B' to B.

In a different direction, we can weaken our requirement to a pair of opposing maps 'to'. In this case, we do, in fact, get a counterpart of the CSB theorem. However, a map 'to' is incredibly close to an isomorphism, differing only via intraclass noninjectivity. I.e., we're starting with a pair of almost-isomorphisms to prove the existence of an isomorphism. The utility of such a result is dubious. Note that we are *not* saying that the opposing maps themselves must be isomorphisms (as in the case of opposing coarsening morphisms). The following example illustrates this.

Ex. Let $S = S' = N$ and let $B = B' = \{(N)\}$, the trivial partition. Define a noninjective, surjective $f_1 : S \to S'$ via $f_1(n) = [(n+1)/2]$ (i.e. $f_1(1) = f_1(2) = 1$ and $f_1(3) = f_1(4) = 2$, etc). Define f_2 the same way, but going from S' to S. Then f_1 and f_2 form a pair of opposing maps 'to', but neither is an isomorphism. An isomorphism does exist, however. Ex. $h(n) = n$.

Prop 2.30: Let f_1 be a map from B 'to' B' and f_2 be a map from B' 'to' B. Then there exists an isomorphism from B to B'.

Pf: We'll require the axiom of choice for this proof. Given $f_1 : S \to S'$ and $f_2 : S' \to S$ as opposing maps 'to', we have that $g_1 = f_1'|_B$ is bijective to B' and $g_2 = f_2'|_{B'}$ is bijective to B. [Note that they aren't inverses in general]. Since we have a pair of opposing surjections, the CSB tells us that $|S| = |S'|$, and we know from the existence of the bijections g_1 and g_2 that $|B| = |B'|$. If f_1 or f_2 is bijective, then we have an isomorphism, since a bijective 'map to' is an isomorphism. However, the existence of a bijection alone does not mean that there exists a bijective 'map to'. If there exists a class-size preserving bijection $k : B \to B'$ (i.e. s.t. $|b| = |k(b)|$ for all b), then proposition 2.22 tells us that an isomorphism exists. Lemma 2.31 below gives us such a k. Let C be the set of nonzero cardinals (possibly capped to avoid having to deal with very large cardinals beyond the $\beth_n$'s). C is strictly linearly ordered and has minimum element 1. Define $X = B$ and $Y = B'$ and $v(b) \equiv |b|$ and $w(b') \equiv |b'|$ and $f = g_1$ and $g = g_2$. Since f_1 and f_2 are surjective, we know that $|g_1(b)| \leq |b|$ and $|g_2(b')| \leq |b'|$, so $w(f(b)) \leq v(b)$ and $v(g(b')) \leq w(b')$. I.e., we have the setup required for lemma 2.31, which then tells us that both g_1 and g_2 can serve as k. Note that this does *not* tell us that f_1 and f_2 are bijective (and thus isomorphisms). However, we can construct an isomorphism from either of their induced g's via proposition 2.22 (which is where the axiom of choice comes in).

Lemma 2.31: Let C denote a (possibly uncountable) strictly linearly-ordered set with a minimum element, and let X and Y be sets s.t. $|X| = |Y|$. Let $v : X \to C$ and $w : Y \to C$. If there exist bijections $f : X \to Y$ and $g : Y \to X$ (not necessarily inverses) s.t. $w(f(x)) \leq v(x)$ for all $x \in X$ and $v(g(y)) \leq w(y)$ for all $y \in Y$, then f satisfies $v(x) = w(f(x))$ for all $x \in X$ and g satisfies $v(g(y)) = w(y)$ for all $y \in Y$.

Pf: Sort the elements of X so that they are in order of increasing C (i.e. pull back the strict linear order on C along V to a (possibly weak) linear order on X). Do the same for Y. Suppose that our proposition holds up to (but not yet including) cardinality c. I.e., for each cardinality $c' < c$, f and g induce bijections between $v^{-1}(c')$ and $w^{-1}(c')$. Now, consider c. For any x s.t. $v(x) = c$, $w(f(x)) \leq c$. Suppose $w(f(x)) = c'$ for some $c' < c$. We already know that $f|_{v^{-1}(c')}$ is a bijection to $w^{-1}(c')$. I.e. f takes some x' with $v(x') = c'$ to $f(x)$. Since $v(x) \neq v(x')$, $x \neq x'$. Since f is bijective, $f(x) \neq f(x')$ contradicting our assumption. Therefore, we must have $f(x) \in w^{-1}(c)$. Since f is a bijection, each x must have a counterpart in $w^{-1}(c)$. This restricted map $f|_{v^{-1}(c)} : v^{-1}(c) \to w^{-1}(c)$ is injective. The same argument holds the other way using g, so we have two opposing injective maps and the CSB theorem tells us that a bijection exists between $v^{-1}(c)$ and $w^{-1}(c)$. However, we have not yet demonstrated that $f|_{v^{-1}(c)}$ is surjective to $w^{-1}(c)$. It's still possible that f maps some x with $v(x) > c$ to some $y \in w^{-1}(c)$. Let's briefly change tack. $g \circ f$ is a bijection from $X \to X$. Moreover, $v(g(f(x))) \leq w(f(x)) \leq v(x)$. I.e., $g \circ f$ takes an x with $v(x)$ to an x' with $v(x') \leq v(x)$. In terms of v, it is a downward-facing bijection. Consider $c = 1$. There is no 'downward', so $g \circ f$ is a bijection from $v^{-1}(c)$ to itself. This means that $f|_{v^{-1}(c)}$ is injective and that g restricted to Im $f|_{v^{-1}(c)} \subseteq w^{-1}(c)$ is surjective to $v^{-1}(c)$. However, we already know that g is injective, so any restriction is injective. This makes the restriction a bijection, which forces f's restriction to be a bijection. I.e., $f|_{v^{-1}(c)}$ is a bijection to $w^{-1}(c)$ and $g|_{w^{-1}(c)}$ is a bijection to $v^{-1}(c)$. Now, suppose our proposition holds up to (but not yet including) some c. The same reasoning tells us that f can't map any element of $v^{-1}(c)$ to an element $y \in Y$ with $w(y) < c$ because that element already has an f^{-1} with $v(f^{-1}(y)) = w(y) < c$. I.e., all the elements with $v(x)$ or $w(y)$ less than c have already been accounted for in f, f^{-1}, g, and g^{-1}. Once again, we have that $(g \circ f)|_{v^{-1}(c)}$ is bijective to $v^{-1}(c)$, and once again this implies that $f|_{v^{-1}(c)}$ is injective and that g restricted to Im $f|_{v^{-1}(c)}$ is surjective to $v^{-1}(c)$. Since g is injective, its restriction is too, and thus is bijective to $v^{-1}(c)$. This forces $f|_{v^{-1}(c)}$ to be bijective to $w^{-1}(c)$ so that the restriction of $g \circ f$ can be bijective. $f|_{v^{-1}(c)}$ therefore is a bijection to $w^{-1}(c)$ and $g|_{w^{-1}(c)}$ is a bijection to $v^{-1}(c)$. By induction, this gives us that f and g satisfy the requisite equalities.

Another approach is to weaken the outcome. Instead of demanding isomorphism, what if we settle for a map 'to'? I.e., let's ask whether a pair of opposing morphisms at least implies that there exists a 'map to' B' from B. We do, in fact, have a version of the CSB in this case.

Prop 2.32: If there exist morphisms (not necessarily surjective) from B to B' and from B' to B, then there exists a 'flexible map to' B' from B and a 'flexible map to' B from B'. If, in addition, there exists a bijection k between B and B' s.t. $|b| \geq |k(b)|$ for every $b \in B$, then there exists a 'map to' B' from B.

This last condition is needed to ensure that we can find a way to match up classes so that each class of b can be mapped to its counterpart surjectively. For example, if $|B| = |B'|$, but every class of B has size 1 and every class of B' has size 2, we're out of luck in that regard.

It follows that if there exist morphisms from B to B' and from B' to B and there exists a bijection $k : B \to B'$ s.t. $|b| \geq |k(b)|$ always and a bijection $k' : B' \to B$ s.t. $|b'| \geq |k'(b')|$ always, then there exist opposing maps 'to'. Proposition 2.30 then tells us that an isomorphism exists. I.e., opposing morphisms and (independent of them) opposing k's guarantee the existence of an isomorphism.

Note that we're not requiring a pair of surjective morphisms. If the map 'to' produced by our proposition is bijective, then it is an isomorphism. We've already seen a counterexample, where a pair of surjective morphisms exist between non-isomorphic partitions. This doesn't mean that we can't *sometimes* get an isomorphism, just that it's not guaranteed. If we have a pair of surjective opposing morphisms, then (via the corollary to the CSB), S and S' are bijective. However, this doesn't mean that our map 'to' has to be. Our proposition is, in fact, more flexible than this. It tells us that we can get a map 'to' even from a pair of non-surjective morphisms — as long as there exists some bijection k resulting k is bijective. Ex. if $S = \{1, 2, 3, 4, 5\}$ and $S' = \{1, 2, 3, 4\}$ and $B = \{(1, 2), (3, 4, 5)\}$ and $B' = \{(1, 2), (3, 4)\}$, then we cannot have a pair of opposing surjective morphisms (since there is no surjective map from S' to S). However, $f : (1, 2, 3, 4, 5) \to (1, 2, 3, 4, 4)$ is a map from B 'to' B' since $f'|_B = \{(1, 2), (3, 4)\}$ is bijective.

Pf: We'll use the axiom of choice in this proof. Let f_1 and f_2 be our two morphisms, and let $g_1 : B \to B'$ and $g_2 : B' \to B$ be their induced maps. In all our definitions (including that of morphism), we required a surjective g, even if f is not surjective. We therefore have two opposing surjective maps, and the corollary to the CSB theorem tells us that there exists a bijection between B and B'. Pick (via the axiom of choice) some such bijection $k : B \to B'$. Then we can pick (again via the axiom of choice) maps $h_b : b \to k(b)$ (they need not be injective or surjective). Since B is a partition, we can define a map $f : S \to S'$ via $f|_b \equiv h_b$ for all $b \in B$. By construction, f is a 'flexible map to'. The argument so far is symmetric, so we have flexible maps 'to' in both directions. If a bijection $k : B \to B'$ exists s.t. $|b| \geq |k(b)|$ always, then we can pick this as our k in the first stage. In that case, $|b| \geq |k(b)|$ by construction, and we can choose each h_b to be surjective. The resulting f is surjective, so we have a map from B 'to' B'.

2.9. **Isomorphism classes.** We can construct isomorphism classes of partitions of a given S and isomorphism classes of partitions across sets (i.e. allowing isomorphic pairs (S, B) and (S', B')) and ask whether our concepts are well-defined for such classes. Specifically, do the notions of morphism, refinement, coarsening, meet, and join make sense for such isomorphism classes?

In the discussion to come, we'll be bandying around lots of equivalence relations, partitions, and classes — not just in the sense of our underlying objects being partitions and classes, but in the sense of objects we want to study in their own right, such as a partition of $Par(S)$ into isomorphism classes. Unfortunately, confusing terminology is unavoidable when dealing with partitions and classes both as our bread and butter and as constructions on the sets of partitions themselves (or on sets of maps between partitions). It is important to keep in mind exactly what each set involves. In particular, it is helpful to keep a close eye on which entities are sets of functions (or sets of sets of functions) and which entities are sets of partitions (or sets of sets of partitions).

For convenience, we'll denote by $Bij(S, S')$ the set of bijections from S to S' and by $Surj(S, S')$ the set of surjections from S to S'. Although these are just sets in general, $Bij(S, S)$ is a group in the obvious way under composition.

Unless otherwise stated, the term "isomorphism" will always refer to an isomorphism of partitions. We'll specifically say "group-isomorphism" when we speak of an isomorphism of groups.

2.9.1. *Automorphisms of B.* Let's begin with automorphisms. Given (S, B), let $Aut(B)$ be the set of isomorphisms from B to itself.

Since B is defined on a given S, the S-dependency is implicit in $Aut(B)$.

Bear in mind that $Aut(B)$ is a set of functions $S \to S$, even though we refer to them as isomorphisms from B to B.

An automorphism f must map each class of B to a class of the same size. This means that its induced g is highly constrained.

Consider $S = \{1, 2, 3, 4\}$ and partitions $B_1 = \{(1, 2), (3, 4)\}$ and $B_2 = \{(1), (2, 3, 4)\}$. There are obvious nontrivial automorphisms of B_1 that swap its two classes. However, for B_2 there can be no automorphism that swaps classes. f must be bijective, and therefore its induced h_b's must be bijective. However, there is no h_b that bijectively maps (1) to $(2, 3, 4)$.

In general, automorphisms may mix classes. However, there is a subset of them which only move elements within existing classes. We'll denote this $PAut(B) \subset Aut(B)$. The induced $f'|_B = Id_B$ for any element of $PAut(B)$.

Consider $S = \{1, 2, 3, 4\}$ and $B = \{(1, 2), (3, 4)\}$. Then $f : (1, 2, 3, 4) = (2, 1, 4, 3)$ is an element of $PAut(B)$, and its induced map $f'|_B = Id_B$. On the other hand, $f : (1, 2, 3, 4) = (3, 4, 1, 2)$ is an element of $Aut(B)$, with an induced map $f'|_B$ that swaps the two classes.

Ex. suppose S has 12 elements and B consists of 3 classes of 2 elements each and 2 classes of 3 elements each. The total number of bijections $S \to S$ is $12! = 479,001,600$. Only $(2!)^3 \cdot (3!)^2 = 288$ of these are in $PAut(B)$. Members of $Aut(B)$ correspond to those which map classes of the same cardinality. There are $3! \cdot 2! = 12$ ways to map B to itself and ensure that the cardinality of classes is preserved. For each such permutation, there are $|PAut(B)|$ ways to permute within the classes. So $|Aut(B)| = 12 \cdot 288 = 3456$.

For a finite S, the trivial partition has $|Aut(B)| = |PAut(B)| = |S|!$, since every permutation is an element of $PAut(B)$. The singleton partition has $|Aut(B)| = |S|!$ and $|PAut(B)| = 1$, since every permutation is an element of $Aut(B)$ but only $Id_B \in PAut(B)$. For $|S| = \beth_n$ $(n \geq 0)$, the trivial partition has $|Aut(B)| = |PAut(B)| = |S| = \beth_{n+1}$ (since the number of permutations is of that cardinality, and we're bounded on both sides), and the singleton partition has $|Aut(B)| = \beth_{n+1}$ and $|PAut(B)| = 1$.

Each $f \in Aut(B)$ induces a bijective $g = f'|_B : B \to B$. As we noted, for $f \in PAut(B)$ we have $f'|_B = Id_B$. Define $Aut'(B)$ to be the set of induced maps, and define $\gamma : Aut(B) \to Aut'(B)$ as the projection map which takes each $f \in Aut(B)$ to $f'|_B$.

Prop 2.33: $Aut(B)$, $Aut'(B)$, and $PAut(B)$ are groups under composition. $PAut(B)$ is a normal subgroup of $Aut(B)$, γ is a full (but not faithful) homomorphism from $Aut(B)$ to $Aut'(B)$, $PAut(B) = \ker \gamma$.

In fact, $Aut(B)$ is a subgroup of $Bij(S, S')$, so $PAut(B) \subseteq Aut(B) \subset Bij(S, S')$.

Pf: ($PAut(B)$ is a group): $PAut(B) = \prod_i Perm(b_i)$ (a direct product), where $Perm(b_i)$ denotes the group of permutations of b_i. As a direct product of groups, it composes and inverts component-wise, and the identity Id_S is just the direct product of the identity permutations.

Pf: ($Aut(B)$ is a group): Consider $f_1, f_2 \in Aut(B)$. They are composable isomorphisms, so proposition 2.9 tells us that their composition is an isomorphism from B to B. Proposition 2.8 tells us that their inverses are isomorphisms from B to B. The identity map Id_S trivially is in $Aut(B)$, and it induces Id_B. We've therefore shown that $Aut(B)$ is a group.

Pf: ($PAut(B)$ is a subgroup of $Aut(B)$): It's a group under the same composition, so if it is a subset it must be a subgroup. Consider $f \in PAut(B)$. Since f only moves elements within each class, $f(b)$ as a set (and thus $f'(b)$ as an element) is unchanged for each b. Therefore, f induces Id_B, which trivially makes it an isomorphism from B to B.

Pf: ($Aut'(B)$ is a group and γ is a full homomorphism to it): We pretty much already showed this. By the definition of $Aut'(B)$, its elements are bijections of B induced by elements of $Aut(B)$. I.e., $Aut'(B) = \text{Im } \gamma$. Translating proposition 1.1 into the relevant language: $\gamma(f_1 \circ f_2) = \gamma(f_1) \circ \gamma(f_2)$ and $\gamma(f^{-1}) = (\gamma(f))^{-1}$. We also know that $\gamma(Id_S) = Id_B$. Therefore, $Aut'(B)$ is a group, γ preserves the multiplication on $Aut(B)$, and (since $Aut'(B) = \text{Im } \gamma$) it is surjective. This makes it a full homomorphism. Because it can be many-to-one, it is not faithful.

Pf: ($PAut(B) = \ker \gamma$): By the definition of $PAut(B)$, its elements are those and only those bijective maps which induce Id_B. I.e., $\gamma(PAut(B)) = \{Id_B\}$.

Pf: ($PAut(B) \triangleleft Aut(B)$): Let $k \in PAut(B)$ and $f \in Aut(B)$. To prove normality, we must show that $f \circ k \circ f^{-1} \in PAut(B)$. As a composition of elements of $Aut(B)$, we know it is in $Aut(B)$. Proposition 1.1 tells us that $(f \circ k \circ f^{-1})|'_B = f'|_B \circ k'|_B \circ (f^{-1})'|_B$ and that $(f^{-1})'|_B = f'|_B^{-1}$. We also know that $k'|_B = Id_B$. Therefore, we get $f'|_B \circ Id_B \circ f'|_B^{-1} = Id_B$. An isomorphism which induces Id_B is an element of $PAut(B)$, so $f \circ k \circ f^{-1} \in PAut(B)$.

For the singleton partition, $PAut(B)$ is the trivial group $\{Id_S\}$ and $Aut(B) \approx Aut'(B)$ (i.e. γ is a group-isomorphism). For the trivial partition, $PAut(B) = Aut(B)$ and $Aut'(B) = \{Id_B\}$ is trivial (i.e. γ is the constant map).

Note that neither $PAut(B)$ nor $Aut(B)$ need be normal in $Bij(S, S)$, as the following example shows.

Note that we have to state this for $PAut(B)$ and $Aut(B)$ separately, even though $PAut(B) \triangleleft Aut(B)$, because normality is not transitive. It is possible to have $K \triangleleft G \triangleleft H$, yet fail to have $K \triangleleft H$.

Ex. let $S = \{1, 2, 3, 4\}$ and $B = \{(1, 2, 3), (4)\}$ and $f : (1, 2, 3, 4) \to (3, 2, 1, 4)$. Then $f \in PAut(B)$. Consider $h : (1, 2, 3, 4) \to (4, 1, 2, 3)$. Then $h \in Bij(S, S)$ and $h^{-1} : (1, 2, 3, 4) \to (2, 3, 4, 1)$. $h \circ f \circ h^{-1} : (1, 2, 3, 4) \to (1, 4, 3, 2)$, which isn't an automorphism of B since it mixes classes. I.e., $hfh^{-1} \notin PAut(B)$. Since $Aut(B) = PAut(B)$ for this particular B, this also serves to prove that $hfh^{-1} \notin Aut(B)$.

Since $PAut(B) \triangleleft Aut(B)$, we can define a quotient group Q. To nobody's surprise, this is just $Aut'(B)$. In fact, $Aut(B)$ is just a direct sum of $Aut'(B)$ and $PAut(B)$.

Prop 2.34: (i) $Aut'(B) = Aut(B)/PAut(B)$, with γ the quotient homomorphism, (ii) we have short exact sequence (SES) $Id_S \to PAut(B) \xrightarrow{i} Aut(B) \xrightarrow{\gamma} Aut'(B) \to Id_B$, (iii) this SES right-splits, (iv) $Aut(B) \approx PAut(B) \rtimes_\phi Aut'(B)$ (i.e. $Aut(B)$ is a semidirect product, with the relevant twisting function ϕ defined in the proof below).

$\approx$ denotes group-isomorphic.

Normally, we write a short exact sequence with 1 at either end. However, here it is clearer if we explicitly state the identity elements.

Pf: (i) Since $PAut \triangleleft Aut(B)$, the left and right cosets are equal. Q is the set of these cosets. A given coset consists of $PAut(B) \circ f$ for some $f \in Aut(B)$. Let $k \in PAut(B)$, and consider $k \circ f$. The induced map $(k \circ f)'|_B = k'|_B \circ f'|_B = Id_B \circ f'|_B = f'|_B$. So, all members of a coset have the same f'. Suppose f_1, f_2 have $f_1'|_B = f_2'|_B$. Then $f_1 \circ f_2^{-1}$ has induced map Id_B, which means that $f_1 \circ f_2^{-1} \in PAut(B)$, so $f_1 = k \circ f_2$ and they are in the same coset. Therefore, the cosets are indexed by $f'|_B$. This means that they are bijective with $Aut'(B)$. Since $f_1 \circ f_2$ implies $(f_1 \circ f_2)'|_B = f_1'|_B \circ f_2'|_B$, γ serves as the quotient homomorphism.

Pf: (ii) The SES just states what we already know: $PAut \triangleleft Aut(B)$ and $Q = Aut'(B)$ with γ the quotient homomorphism.

Pf: (iii,iv) (preface): The SES right-splits iff we have a semi-direct product so (iii) and (iv) are equivalent. Setwise, $PAut(B) \rtimes_\phi Aut'(B)$ and $PAut(B) \oplus Aut'(B)$ both are just $PAut(B) \times Aut'(B)$. We'll denote its elements (p, l), with $p \in PAut(B)$ and $l \in Aut'(B)$. Bear in mind that l isn't an arbitrary bijection on B. It must take classes to classes of the same size. The "semi-directness" comes from the algebraic structure on $PAut(B) \rtimes_\phi Aut'(B)$. The "twisting" function ϕ tells us how we deviate in the first component from simple $PAut(B)$ multiplication as we move around $Aut'(B)$. Specifically, $\phi : Aut'(B) \to Aut(PAut(B))$ (where the outer Aut on the right is just the automorphism group of the group $PAut(B)$) is a homomorphism. By ϕ_l, we denote the way that $PAut(B)$ is transformed by element l. The multiplication is then defined as $(p_1, l_1) \cdot (p_2, l_2) \equiv (p_1 \cdot \phi_l(p_2), l_1 \cdot l_2)$, where the multiplication in each component is just that of the component group. Since the multiplications on both $PAut(B)$ and $Aut'(B)$ are function composition, we can write the product as $(p_1 \circ \phi_l(p_2), l_1 \circ l_2)$. Note that we haven't yet defined ϕ. To construct it (and see the need for it), we'll first attempt to build a group-isomorphism from $Aut(B)$ to $PAut(B) \oplus Aut'(B)$, fail, and then choose ϕ to correct the defect.

Pf: (iii,iv) (attempt for direct product): By the well-ordering principle, impose a strict linear ordering on S. There are many ways to do this, but we don't care — just pick one. Let $f \in Aut(B)$. This defines an $l \equiv f'|_B : B \to B$ that takes classes to classes of the same size. There exists a unique $\widetilde{l} \in Aut(B)$ s.t. $\widetilde{l}'|_B = l$ *and* which preserves the order within each class. I.e., within a given b, $\widetilde{l}(x) < \widetilde{l}(y)$ iff $x < y$. Obviously, it doesn't preserve order *between* classes as well unless $l = Id_B$. $\widetilde{l}$ is unique on each b (and thus overall) because $|b| = |f(b)|$ and the sets of elements are fixed (i.e. b and $f(b)$), so there can be only one $\widetilde{l}|_b$ which preserves the *strict* linear order. We thus have selected a preferred representative of the class $[f] \in Aut'(B)$. Since $\widetilde{l}'|_B = f'|_B$, $f' \circ \widetilde{l}'^{-1} = Id_B$ and $p \equiv f \circ \widetilde{l}^{-1} \in PAut(B)$. We therefore have a map $\beta : Aut(B) \to PAut(B) \oplus Aut'(B)$, given by $\beta(f) \equiv ((f \circ \widetilde{f'|_B}^{-1}), f'|_B)$. We also have $f = p \circ \widetilde{l}$ for the p and l thus defined. β is surjective. Suppose we are given (p, l), with $l \in Aut'(B)$ and $p \in PAut(B)$. Since $l \in Aut'(B)$ is a bijection that preserves class-size, it defines a $\widetilde{l} \in Aut(B)$. We then just take $f = p \circ \widetilde{l}$, and it is trivial to see that $\beta(f) = (p, l)$. β is also injective. Suppose we are given $f_1, f_2 \in Aut(B)$. Let $\beta(f_1) = (p_1, l_1)$ and $\beta(f_2) = (p_2, l_2)$. Suppose that $l_1 = l_2$ and $p_1 = p_2$. Then $f_1 = p_1 \circ \widetilde{l_1}$ and $f_2 = p_2 \circ \widetilde{l_2} = p_1 \circ \widetilde{l_1}$. So $f_1 = f_2$. Therefore, both $l_1 = l_2$ and $p_1 = p_2$ iff $f_1 = f_2$, and we see that β is injective. Next, let's try to show that β is a homomorphism from $Aut(B)$ to $PAut(B) \oplus Aut'(B)$. [Before doing so, we'll observe that $p \circ \widetilde{l} \neq \widetilde{l} \circ p$ in general. Ex. let $S = \{1, 2, 3, 4\}$, let $B = \{(1, 2), (3, 4)\}$, let l swap the two classes, and let $p : (1, 2, 3, 4) \to (2, 1, 3, 4)$. Then $\widetilde{l} : (1, 2, 3, 4) \to (3, 4, 1, 2)$. In this case, $p \circ l : (1, 2, 3, 4) \to (3, 4, 2, 1)$ and $l \circ p : (1, 2, 3, 4) \to (4, 3, 1, 2)$.] It is trivial to see that $\beta(Id_S) = (Id_B, Id_S)$, the identity of $PAut(B) \oplus Aut'(B)$. Let $\beta(f_1) = (p_1, l_1)$ and $\beta(f_2) = (p_2, l_2)$ and $\beta(f_2 \circ f_1) = (p, l)$. Since we've assumed direct-product multiplication, β is only a homomorphism if $p = p_2 \circ p_1$ and $l = l_2 \circ l_1$. Since $(f_2 \circ f_1)' = f_2' \circ f_1'$, it follows that $l = l_2 \circ l_1$, as needed. We then have that $p_1 = f_1 \circ \widetilde{l_1}^{-1}$ and $p_2 = f_2 \circ \widetilde{l_2}^{-1}$ and $p_2 \circ p_1 = f_2 \circ \widetilde{l_2}^{-1} \circ f_1 \circ \widetilde{l_1}^{-1}$. On the other hand, $p = (f_2 \circ f_1) \circ (\widetilde{l_2 \circ l_1})^{-1}$. It is easy to see that $\widetilde{l_2 \circ l_1} = \widetilde{l_2} \circ \widetilde{l_1}$, so $p = f_2 \circ f_1 \circ \widetilde{l_1}^{-1} \circ \widetilde{l_2}^{-1}$, which is not equal to $p_2 \circ p_1$. Therefore, β is not a homomorphism from $Aut(B)$ to $PAut(B) \oplus Aut'(B)$.

Pf: (iii,iv) (correcting the defect): Can ϕ rectify this? We need $p_2 \circ \phi_{l_2}(p_1)$ to equal $f_2 \circ f_1 \circ \widetilde{l_1}^{-1} \circ \widetilde{l_2}^{-1}$. The right side can be written $f_2 \circ p_1 \circ \widetilde{l_2}^{-1}$, and the left side is $f_2 \circ \widetilde{l_2}^{-1} \circ \phi_{l_2}(p_1)$. To make these equal, we need $p_1 \circ \widetilde{l_2}^{-1} = \widetilde{l_2}^{-1} \circ \phi_{l_2}(p_1)$. This is obtained via $\phi_{l_2} \equiv \widetilde{l_2} \circ p_1 \circ \widetilde{l_2}^{-1}$. This is indeed a map from $PAut(B)$ to $PAut(B)$, since it is a composition of three isomorphisms from B to B and $(\widetilde{l_2} \circ p_1 \circ \widetilde{l_2}^{-1})' = Id_B$. To see that ϕ_{l_2} is surjective, suppose that $p \in PAut(B)$, and let $q \equiv \widetilde{l_2}^{-1} \circ p \circ \widetilde{l_2}$. Clearly, $q \in PAut(B)$ and $\phi_{l_2}(q) = p$. To see that ϕ_{l_2} is injective, suppose that $\widetilde{l_2} \circ p_1 \circ \widetilde{l_2}^{-1} = \widetilde{l_2} \circ p_2 \circ \widetilde{l_2}^{-1}$. Since these are all isomorphisms, $p_1 = p_2$. So we have a bijection. $\phi_{l_2}(Id_S) = Id_S$, and $\phi_{l_2}(p_2 \circ p_1) = \widetilde{l_2} \circ p_2 \circ \widetilde{l_2}^{-1} \circ \widetilde{l_2} \circ p_1 \circ \widetilde{l_2}^{-1} = \widetilde{l_2} \circ p_2 \circ p_1 \circ \widetilde{l_2}^{-1} = \phi_{l_2}(p_2 \circ p_1)$. A bijective homomorphism is a group-isomorphism, so $\phi_{l_2} \in Aut(PAut(B))$. Next, we must show that $\phi : Aut'(B) \to Aut(PAut(B))$ itself is a homomorphism. $\phi_{Id_B}(p) = \widetilde{Id_B} \circ p \circ \widetilde{Id_B}^{-1} = Id_S \circ p \circ Id_S = p$. So $\phi_{Id_B} = Id_{PAut(B)}$. Let $l_1, l_2 \in Aut'(B)$. Then $\phi_{l_2 \circ l_1}(p) = \widetilde{l_2 \circ l_1} \circ p \circ \widetilde{l_2 \circ l_1}^{-1} = \widetilde{l_2} \circ \widetilde{l_1} \circ p \circ \widetilde{l_1}^{-1} \circ \widetilde{l_2}^{-1} = \phi_{l_2}(\phi_{l_1}(p))$. Since composition is the multiplication on $Aut(PAut(B))$, ϕ is a homomorphism from $Aut'(B)$ to $Aut(PAut(B))$. This corrects our defect, and we have a semidirect product rather than a general group extension. The relevant ϕ is given by $\phi_l(p) \equiv \widetilde{l} \circ p \circ \widetilde{l}^{-1}$. By construction, our β is a homomorphism from $Aut(B)$ to $PAut(B) \rtimes_\phi Aut'(B)$ (with the latter endowed with this multiplication). It is a bijective homomorphism and thus a group-isomorphism, establishing (iii).

Pf: (iii,iv) (right-splitting): For the SES to right-split, we need a homomorphism $\alpha : Aut'(B) \to Aut(B)$ such that $\gamma \circ \alpha = Id_{Aut'(B)}$. This is easy to construct. In fact, we've already have constructed it. Define $\alpha(l) \equiv \widetilde{l}$ using our choice of strict linear order on S and the method described earlier. It is not hard to see that $\alpha(Id_B) = Id_S$. Since our $\widetilde{foo}$ operator (now called α) preserves the strict linear order, and since $Id_B(b) = b$, α must map each element to itself. Next, consider $\alpha(l_2 \circ l_1)$. Once again, we must preserve our strict linear order at each step and thus in the composition. There is only one way to do so, and we have $\alpha(l_2 \circ l_1) = \alpha(l_2) \circ \alpha(l_1)$. We thus have a homomorphism. Consider $\gamma(\alpha(l))$. This equals $\widetilde{l}' = l$, so we get $\gamma \circ \alpha = Id_{Aut'(B)}$. Note that we must use the same strict linear order (i.e. the same $\widetilde{l}$ definition) in both the semi-direct product and the α homomorphism. Different choices of linear order yield different ϕ's, β's and α's — but we don't care. As long as we define them all in tandem, they play well together. The fact that we can do this in different ways is immaterial. There can be many ways to skin this horse. Also note that, like the choice of strict linear order, we have a choice of whether to define $f = p \circ \widehat{l}$ or $f = \widehat{l} \circ p$. Selecting the latter works just as well, with appropriate modifications to the rest of the definitions.

The Aut, $PAut$, and Aut' groups of isomorphic B's are group-isomorphic, as the following proposition demonstrates.

Prop 2.35: If (S,B) and (S',B') are isomorphic, then $Aut(B) \approx Aut(B')$, $PAut(B) \approx PAut(B')$, and $Aut'(B) \approx Aut'(B')$. Any choice of isomorphism $f : S \to S'$ yields a specific group-isomorphism for each of these.

Note that distinct f's can yield the same group-isomorphisms for these. There isn't a one-to-one correspondence between partition-isomorphisms from B to B' and the associated group-isomorphisms between $Aut(B)$ and $Aut(B')$, etc.

Pf: (Aut): Let $f : S \to S'$ be an isomorphism from B to B'. Define a map $\alpha : Aut(B) \to Aut(B')$ via $\alpha(k) \equiv f \circ k \circ f^{-1}$ for every $k \in Aut(B)$. This is injective because $\alpha(k_1) = \alpha(k_2)$ implies $f \circ k_1 \circ f^{-1} = f \circ k_2 \circ f^{-1}$. Since these are all isomorphisms, we can compose f on the right and f^{-1} on the left to get $k_1 = k_2$. α is surjective because if $l \in Aut_{B'}(S')$ then $k \equiv f^{-1} \circ l \circ f \in Aut_B(S)$ and $\alpha(k) = l$. So α is bijective. As the composition of three isomorphisms, $\alpha(k)$ is an automorphism of B'. To see that α is a group-isomorphism, note that (i) $\alpha(Id_S) = Id_{S'}$, (ii) $\alpha(k^{-1}) = f \circ k^{-1} \circ f^{-1} = (f \circ k \circ f^{-1})^{-1} = (\alpha(k))^{-1}$, and (iii) given $k_1, k_2 \in Aut_B(S)$, $\alpha(k_1 \circ k_2) = f \circ k_1 \circ f^{-1} \circ f \circ k_2 \circ f^{-1} = f \circ (k_1 \circ k_2) \circ f^{-1} = \alpha(k_1 \circ k_2)$. We have a bijective homomorphism, and any bijective homomorphism is a group-isomorphism.

Pf: $(PAut)$: Consider the α we just defined, and let $k \in PAut(B)$. Since we're dealing with isomorphisms, the induced g is just $k'|_B$ and is bijective to B'. By proposition 1.1, $(k_1 \circ k_2)' = k_1' \circ k_2'$ and $(k')^{-1} = (k^{-1})'$. So the induced maps compose and invert as expected. Therefore, $f \circ k \circ f^{-1}$ has induced map $f' \circ k' \circ f'^{-1}$. However, for $k \in PAut(B)$, the induced $k'|_B = Id_B$ by definition. Therefore, $(f' \circ k' \circ f'^{-1})|_{B'} = (f' \circ f'^{-1})|_{B'} = Id_{B'}$, and $\alpha(k) \in PAut(B')$. We thus have shown that $\alpha|_{PAut(B)} : PAut(B) \to PAut(B')$. We already know that α is injective, so its restriction is injective as well. Consider some $l \in PAut(B')$. Define $k \equiv f^{-1} \circ l \circ f$. Clearly, $\alpha(k) = l$. By the same reasoning as before, this yields $k'|_B = Id_B$ as the partition map. Therefore, $\alpha|_{PAut(B)}$ is surjective (and thus bijective) to $PAut(B')$. As a bijective homomorphism, it is a group-isomorphism.

Pf: (Aut'): Since $Aut(B) \approx Aut(B')$ and $PAut(B) \approx PAut(B')$, it immediately follows from group theory that $Aut(B)/PAut(B) \approx Aut(B')/PAut(B')$. Therefore $Aut'(B) \approx Aut'(B')$. However, let's show this explicitly. Given isomorphism f, define a map between $Aut'(B)$ and $Aut'(B')$ via $\alpha'([k]) \equiv [f \circ k \circ f^{-1}]$, with the [] denoting classes of elements of $Aut()$ that have the same induced map. Since f is an isomorphism, for $k \in Aut(B)$ we have that $(f' \circ k' \circ f'^{-1})|_{B'} = f'|_B \circ k'|_B \circ f^{-1}|_{B'}$. Given $k_1, k_2 \in Aut(B)$, $k_1'|_B = k_2'|_B$ iff $(f' \circ k_1' \circ f'^{-1})|_{B'} = (f' \circ k_2' \circ f'^{-1})|_{B'}$. I.e., α' is well-defined and a bijection between the classes of induced maps. Define $[k_1] \circ [k_2] \equiv [k_1 \circ k_2]$, which clearly is well-defined since $f' \circ k_1' \circ f'^{-1} \circ f' \circ k_2' \circ f'^{-1} = f' \circ (k_1' \circ k_2') \circ f'^{-1}$ and is independent of the choice of representatives. We also immediately see that $[Id_S] = [Id_{S'}]$, and that $[k^{-1}] = [k]^{-1}$ since $(f' \circ k'^{-1} \circ f'^{-1}) = (f' \circ k' \circ f'^{-1})^{-1}$ and is independent of the representative. We have a bijective homomorphism, and thus a group-isomorphism, between $Aut'(B)$ and $Aut'(B')$.

Any two isomorphisms between B and B' are related by an automorphism of B or, equivalently, of B'. The following two propositions formalize this.

Prop 2.36: If (S,B) and (S',B') are isomorphic, then any two isomorphisms f_1 and f_2 between them are related by $f_2 = f_1 \circ h$ for some $h \in Aut(B)$ and (equivalently) $f_2 = k \circ f_1$ for some $k \in Aut(B')$.

Pf: $h \equiv f_1^{-1} \circ f_2$ is the composition of two isomorphisms, and thus is an isomorphism. It clearly is from (S,B) to (S,B), and therefore is an element of $Aut(B)$. Ditto on the other side using $k \equiv f_2 \circ f_1^{-1}$ to get an element of $Aut(B')$.

It follows that from any isomorphism f, we can obtain all other isomorphisms $(S,B) \to (S',B')$ as $f \circ Aut(B)$ or (equivalently) $Aut(B') \circ f$, to play fast and loose with notation. Note that allowing automorphisms at both ends buys us no new isomorphisms. I.e. as sets, $f \circ Aut(B) = Aut(B') \circ f = Aut(B') \circ f \circ Aut(B)$.

This also tells us that for any $h \in Aut(B)$, $f = k \circ f \circ h$ for some $k \in Aut(B')$ and for any $k \in Aut(B')$, $f = k \circ f \circ h$ for some $h \in Aut(B)$.

Prop 2.37: If (S,B) and (S',B') are isomorphic, then any two isomorphisms f_1 and f_2 between them that yield the same induced map $f_1'|_B = f_2'|_B$ are related by $f_2 = f_1 \circ h$, where $h \in PAut(B)$ or (equivalently) $f_2 = k \circ f_1$, where $k \in PAut(B')$.

We already know that f_1 and f_2 are related by an element of $Aut(B)$ (or $Aut(B')$), but if their induced maps are equal then the element is of $PAut(B)$ (or $PAut(B')$).

Pf: By prop 2.36, they are related by an h or k in the relevant Aut group. However, since those necessarily leave the partition map the same, they must induce Id_B or $Id_{B'}$. I.e., they are members of $PAut(B)$ and $PAut(B')$.

2.9.2. *Isomorphisms of B and B'.* Our discussion of automorphisms focused on a specific (S, B). Let's now expand our scope to isomorphisms between different partitions. There's no need to stick with the same S anymore, so we'll consider isomorphisms between (S, B) and (S', B').

We'll denote by $Iso(B, B')$ the set of isomorphisms between B and B'. Bear in mind that its elements are bijective maps between S and S', not maps between B and B'.

As with $Aut(B)$, the dependence on the underlying sets (both S and S' in this case) is implicit.

$Iso(B, B') \subseteq Bij(S, S')$, but the two needn't be equal. They are equal iff B and B' are the trivial partitions of their respective sets.

Each isomorphism f induces a bijective map $g = f'|_B$ from B to B'. Just as we defined $Aut'(B)$ to be the set of maps induced by automorphisms, we'll define $Iso'(B, B')$ to be the set of maps induced by isomorphisms. As such, there is a map $\gamma_I : Iso(B, B') \to Iso'(B, B')$ that takes each $f \in Iso(B, B')$ to the induced map $g = f'|_B \in Iso'(B, B')$. There is no counterpart of $PAut(B)$ when it comes to general isomorphisms because there is no notion of an identity map between B and B'. We cannot speak of isomorphisms as "swapping classes" or "only rearranging elements within classes", because we're dealing with two distinct partitions.

$Iso(B, B')$ and $Iso'(B, B')$ are not groups, since their elements are not composable unless $B = B'$ (in which case, we're just dealing with $Aut(B)$ and $Aut'(B)$). We can only compose $f_1 \in Iso(B, B')$ with $f_2 \in Iso(B', B'')$ to get an element $f_2 \circ f_1 \in Iso(B, B'')$. We also have no identity. What would it mean to be the "identity isomorphism" between B and B'? Since $Iso(B, B')$ and $Iso'(B, B')$ are not groups, γ_I can't be a homomorphism. It's just a surjective map of sets.

The closest we can get to a group is a category of (S, B) pairs, perhaps with the requirement that $S \subseteq U$ for some "universe" U to avoid thorny set-theoretic issues. Depending on our preference, the arrows could be morphisms or surjective morphisms. We won't delve into this or other category-theoretic constructs relating to partitions.

What do the classes of $Iso'(B, B')$ look like? Suppose f_1 and f_2 have induced maps $g_1 = g_2$. Since all these maps are bijective, f_1 and f_2 can only differ by a permutation of the way elements are mapped intraclass. I.e., $f_2|_b = f_1|_b \circ p_b$ for some permutation p_b of b. Equivalently it equals $q_b \circ f_1|_b$ for some permutation q_b of $f_2(b)$. We require some such p_b (or q_b) for each b, and can stitch these into a bijection $p : S \to S$ that takes B to itself and only shuffles elements within each class of B. However, we have a name for just such a map. Since $p' = Id_B$, it's a member of $PAut(B)$ (or, for q, a member of $PAut(B')$). Conceptually, $Iso'(B, B')$ 'looks like' $Iso(B, B')/PAut(B)$ (or, equivalently, $Iso(B, B')/PAut(B')$). However, these aren't valid expressions. $Iso(B, B')$ isn't a group, and $PAut(B)$ isn't a normal subgroup of it.

Since $Iso(B, B')$ is just a set, the best we can do is construct an equivalence relation that implements the desired effect via a quotient set. Let $f_1 \sim_p f_2$ iff $f_2 = f_1 \circ p$ for some $p \in PAut(B)$. Similarly, let $f_1 \sim_q f_2$ iff $f_2 = q \circ f_1$ for some $q \in PAut(B')$. For completeness, let $f_1 \sim_{pq} f_2$ iff $f_2 = q \circ f_1 \circ p$ for some $q \in PAut(B')$ and $p \in PAut(B)$.

These are equivalence relations because $PAut(B)$ and $PAut(B')$ are groups. $f_1 \sim_p f_1$ via $p = Id_S$. If $f_1 \sim_p f_2$ via some p, then $f_2 \sim_p f_1$ via p^{-1}. If $f_1 \sim_p f_2$ via p_1 and $f_2 \sim_p f_3$ via p_2, then $f_3 = f_2 \circ p_2 = f_1 \circ p_1 \circ p_2$, so $f_1 \sim_p f_3$ via $p_1 \circ p_2$. The same holds on the other side for $\sim_q$ and on both sides for sim_{pq}.

Bear in mind that $\sim_p$ and $\sim_q$ and $\sim_{pq}$ all depend on both B and B' (since they are equivalence relations on $Iso(B, B')$), even though the notation doesn't reflect it.

Prop 2.38: All three of $\sim_p$, $\sim_q$, and $\sim_{pq}$ are the same. They partition $Iso(B,B')$ in the same way as $Iso'(B,B')$. I.e., they partition it by induced $f'|_B$.

Pf: ($\sim_p \rightarrow Iso'$): Suppose $f_2 = f_1 \circ p$ for $p \in PAut(B)$. Then $f_2'|_B = f_1'|_B \circ p'$. However, $p' = Id_B$. So $f_2'|_B = f_1'|_B$, and we're in the same f'-class. ($Iso' \rightarrow \sim_p$): Suppose $f_1'|_B = f_2'|_B$. We saw in proposition 2.37 that $f_2 = f_1 \circ p$ for some $p \in PAut(B)$. ($\sim_q \leftrightarrow Iso'$): The same exact arguments (and proposition) apply to $\sim_q$. ($\sim_{pq} \rightarrow Iso'$): In this direction, the same argument holds. If $f_2 = q \circ f_1 \circ p$, then $f_2'|_B = Id_{B'} \circ f_1'|_B \circ Id_B = f_1'|_B$, so we're in the same f'-class. ($Iso' \rightarrow \sim_{pq}$): Suppose $f_1'|_B = f_2'|_B$. We know (again from proposition 2.37) that $f_2 = f_1 \circ p$ for some $p \in PAut(B)$. This means that $f_1 \sim_{pq} f_2$ using $q = Id_{S'}$. Note that we equally well could have used q with $p = Id_S$ instead.

Since $Iso'(B,B')$ embodies our concept of 'looks like $Iso(B,B')/PAut(B)$', we won't bother worrying about $\sim_p$ or $\sim_q$ or $\sim_{pq}$ or any of the quotients. We defined Iso' in terms of $f'|_B$ values rather than classes of functions. However, we now know that the distinction is irrelevant. The classes of $Iso(B,B')/\sim_p$ are identical to the f'-classes. We can consider the elements of $Iso'(B,B')$ to be either the $f'|_B$ functions themselves or the f'-classes of $Iso(B,B')$. We'll use $Iso'(B,B')$ flexibly, and treat it's elements as functions from B to B' or classes of functions from S to S' as needed.

As with $Iso(B,B')$, we'll take some notational liberties and write $Iso'(B,B')^{-1}$ to denote $\{f'|_{B'}^{-1}; f'|_B \in Iso'(B,B')\}$ and $Iso'(B',B'') \circ Iso'(B,B')$ to denote $\{f_2'|_B \circ f_1'|_B; f_1'|_B \in Iso'(B,B'), f_2'|_B \in Iso'(B',B'')\}$.

Prop 2.39: As statements about sets: (i) $Iso(B,B')^{-1} = Iso(B',B)$. (ii) $Iso(B',B'') \circ Iso(B,B') = Iso(B,B'')$. (iii) If B is isomorphic to B_1 and B' is isomorphic to B_1', then $|Iso(B,B')| = |Iso(B_1,B_1')|$, and any specific pair of isomorphisms established a specific bijection between $Iso(B,B')$ and $Iso(B_1,B_1')$. (iv) $Iso(B,B') \circ Aut(B) = Aut(B') \circ Iso(B,B') = Iso(B,B')$. (v) If B and B' are isomorphic, then $|Iso(B,B')| = |Iso(B',B)| = |Aut(B)| = |Aut(B')|$.

We need the caveat in (v) because if B is not isomorphic to B' then $Iso(B,B') = \emptyset$, but $Aut(B)$ and $Aut(B')$ (which need not be group-isomorphic to one another in this case) are not empty.

Pf: (i) f is an isomorphism from B to B' iff f^{-1} is an isomorphism from B' to B. (ii) Isomorphisms compose, so $Iso(B',B'') \circ Iso(B,B') \subseteq Iso(B,B'')$. Given any $f \in Iso(B,B'')$, pick any $f_1 \in Iso(B,B')$, and let $f_2 \equiv f \circ f_1^{-1}$. Then $f_2 \in Iso(B',B'')$ and $f = f_2 \circ f_1$. So $Iso(B,B'') \subseteq Iso(B',B'') \circ Iso(B,B')$. (iii) Pick isomorphisms $f_1 : B \rightarrow B_1$ and $f_2 : B' \rightarrow B_1'$. Then $k \in Iso(B,B')$ iff $f_2 \circ k \circ f_1^{-1} \in Iso(B_1,B_1')$. Moreover, the map $k \rightarrow f_2 \circ k \circ f_1^{-1}$ is invertible and thus establishes a bijection between $Iso(B,B')$ and $Iso(B_1,B_1')$. (iv) $Iso(B,B') \circ Id_S = Iso(B,B')$, so $Iso(B,B') \subseteq Iso(B,B') \circ Aut(B)$. Consider $f \circ p$ with $f \in Iso(B,B')$ and $p \in Aut(B)$. The composition is an isomorphism $f \circ p \in Iso(B,B')$, so $Iso(B,B') \circ Aut(B) \subseteq Iso(B,B')$. The same argument applies on the other side with $Aut(B')$. (v) Suppose B and B' are isomorphic. $f \in Iso(B,B')$ iff $f^{-1} \in Iso(B',B)$, so the map $f \rightarrow f^{-1}$ is invertible and establishes a bijection between $Iso(B,B')$ and $Iso(B',B)$. We already saw in proposition 2.35 that $|Aut(B)| = |Aut(B')|$ (and, in fact, they are group-isomorphic) when B and B' are isomorphic. All that remains is to exhibit a bijection between $Iso(B,B')$ and $Aut(B)$. Pick some $f_0 \in Iso(B,B')$. We already know from proposition 2.36 that any $f \in Iso(B,B')$ satisfies $f = f_0 \circ k$ for some $k \in Aut(B)$. We thus have a map from $Iso(B,B')$ to $Aut(B)$, given by $f \rightarrow f_0^{-1} \circ f$. This map is patently invertible and thus establishes the desired bijection.

We also have a corresponding proposition about our quotient Iso' sets.

Prop 2.40: As statements about sets: (i) $Iso'(B,B')^{-1} = Iso'(B',B)$. (ii) $Iso'(B',B'') \circ Iso'(B,B') = Iso'(B,B'')$. (iii) If B is isomorphic to B_1 and B' is isomorphic to B_1', then $|Iso'(B,B')| = |Iso'(B_1,B_1')|$, and any specific pair of isomorphisms established a specific bijection between $Iso'(B,B')$ and $Iso'(B_1,B_1')$. (iv) $Iso'(B,B') \circ Aut'(B) = Aut'(B') \circ Iso'(B,B') = Iso'(B,B')$. (v) If B and B' are isomorphic, then

$|Iso'(B,B')| = |Iso'(B',B)| = |Aut'(B)| = |Aut'(B')|$.

Pf: Solely for the purposes of this proof, denote by $[f]$ an f'-class of $Iso(B,B')$ (i.e. an element of $Iso'(B,B')$ viewed as a set of isomorphisms). (i) Since $f'^{-1} = (f^{-1})'$, $f_1'|_B = f_2'|_B$ iff $(f_1^{-1})|'_{B'} = (f_2^{-1})|'_{B'}$. This means that $[f^{-1}] = \{f^{-1}; f \in [f]\}$. (ii) Since induced maps compose, $f_2' \circ f_1' = (f_2 \circ f_1)'$. Let $f_1 \in Iso(B,B')$ and $f_2 \in Iso(B',B'')$, which implies that $f_1'|_B \in Iso'(B,B')$ and $f_2'|_{B'} \in Iso'(B',B'')$. Then $f_2 \circ f_1 \in Iso(B,B'')$, and thus $f_2'|_{B'} \circ f_1'|_B \in Iso'(B,B'')$. Now, suppose that $f'|_B \in Iso'(B,B'')$, meaning there is some $f \in Iso(B,B'')$ which induces that $f'|_B$. As in our proof of proposition 2.39, pick any $f_1 \in Iso(B,B')$ and let $f_2 \equiv f \circ f_1^{-1}$. Then $f_2 \in Iso(B',B'')$ and $f = f_2 \circ f_1$, so $f_2'|_{B'} \in Iso'(B',B'')$ and $f_1'|_B \in Iso'(B,B')$ and $f'|_B = f_2'|_{B'} \circ f_1'|_B$. (iii) As in our proof of proposition 2.39, pick isomorphisms $f_1 : B \to B_1$ and $f_2 : B' \to B_1'$. We then have the bijection from $Iso(B,B')$ to $Iso(B_1,B_1')$ given by $k \to f_2 \circ k \circ f_1^{-1}$, which induces a map from $Iso'(B,B')$ to $Iso'(B_1,B_1')$ given by $k'|_B \to (f_2' \circ k' \circ (f_1')^{-1})|_{B_1}$. Since $f_1'|_B \in Iso'(B,B_1)$ and $f_2'|_{B'} \in Iso'(B',B_1')$ and $k'|_B \in Iso'(B,B')$, we have $(f_2' \circ k' \circ (f_1')^{-1})|_{B_1} \in Iso'(B_1,B_1')$. Moreover, the map $k'|_B \to (f_2' \circ k' \circ (f_1')^{-1})|_{B_1}$ is patently invertible and thus establishes the desired bijection. (iv) As in our proof of proposition 2.39, $Iso(B,B') \circ Id_S = Iso(B,B')$, which induces $Iso'(B,B') \circ Id_B = Iso'(B,B')$. So $Iso'(B,B') \subseteq Iso'(B,B') \circ Aut'(B)$. Consider $f' \circ p'$ with $f' \in Iso'(B,B')$ and $p' \in Aut'(B)$. Since $f' \in Iso'(B,B')$, it is induced by some $f \in Iso(B,B')$. Similarly, p' is induced by some $p \in Aut(B)$. The composition $f \circ p \in Iso(B,B')$, so $f' \circ p' \in Iso'(B,B')$. Therefore $Iso'(B,B') \circ Aut'(B) \subseteq Iso'(B,B')$. The same argument applies on the other side with $Aut'(B')$. (v) Suppose B and B' are isomorphic. We saw that $f \to f^{-1}$ is a bijection between $Iso(B,B')$ and $Iso(B',B)$, and it induces the bijection $f'|_B \to (f')^{-1}|_{B'}$ (since $(f')^{-1} = (f^{-1})'$) between $Iso'(B,B')$ and $Iso'(B',B)$. We know from proposition 2.35 that $|Aut'(B)| = |Aut'(B')|$ (and that they are in fact group-isomorphic). All that remains is to exhibit a bijection between $Iso'(B,B')$ and $Aut'(B)$. Pick some $f_0 \in Iso(B,B')$. In proposition 2.39, we established a bijection $f \to f_0^{-1} \circ f$ between $Iso(B,B')$ and $Aut(B)$. This induces a bijection $f'|_B \to ((f_0')^{-1} \circ f)|_B$ between $Iso'(B,B')$ and $Aut'(B)$.

2.10. **Isomorphism classes of partitions.** We've discussed automorphisms of a given B and isomorphisms between a given B and B'. Let's now do something different and consider sets of partitions rather than sets of functions. I.e. we'll work with subsets of $Par(S)$ (or perhaps $CPar(S)$).

As mentioned, the language can be confusing here. Partitions are made of classes, we talk about classes of maps between partitions, and now we're talking about classes *of* partitions. Unfortunately, there's no way around this without introducing a lot of peculiar homegrown jargon. We simply need to keep in mind what objects and maps are under consideration.

2.10.1. *Isomorphism classes on a given S.* Let's start by defining an equivalence relation on $Par(S)$ as $B_1 \sim B_2$ (both on S) iff B_1 is isomorphic to B_2. This defines a partition of $Par(S)$ into isomorphism classes.

Since isomorphisms compose and are invertible, and Id_S is an isomorphism from B to itself, this is indeed an equivalence relation. As such, it partitions $Par(S)$ into equivalence classes.

A countable partition cannot be isomorphic to an uncountable one, because $f'|_B$ must be bijective for an isomorphism. Therefore, the subset $CPar(S) \subseteq Par(S)$ is a union of whole isomorphism classes (i.e. there is no isomorphism class with members both inside and outside $CPar(S)$). It therefore is meaningful to restrict $\sim$ to $CPar(S)$, and the resulting partition of $CPar(S)$ is a subset of the partition that $\sim$ induces on $Par(S)$.

Loosely in keeping with our convention of 'priming' quotient sets, we'll define $Par'(S) \equiv Par(S)/\sim$ and and $CPar'(S) \equiv CPar(S)/\sim$ to be the corresponding sets of isomorphism classes. We just saw that $CPar'(S) \subseteq Par'(S)$.

We write $CPar(S) \subseteq Par(S)$ and $CPar'(S) \subseteq Par'(S)$, but we have = (in both cases) iff S is countable. If S is uncountable, both must be proper subsets.

As a general rule, we tend to only care about objects modulo isomorphism. For that to be the case here, we must ask whether our notion of isomorphism class is compatible with other important concepts relating to partitions of sets. For convenience, we'll denote the isomorphism class of B by $[B]$.

As usual, the relevant S is implicit in this notation.

Before considering this important question, let's extend our notion of isomorphism class to include partitions of different S's rather than only of the same S.

2.10.2. *Isomorphism classes across S's.* To avoid certain thorny set theory issues, we'll assume that all our S's are subsets of some fixed "universe" U in which we work.

Define $Par()$ to be all pairs (S, B) with $S \subseteq U$. I.e., all partitions of all subsets of U. We'll define $CPar()$ in the same way as all countable partitions of all (possibly uncountable) subsets of U.

Our existing $\sim$, defined on each S independently, readily extends to $Par() = \cup_{S \subseteq U} Par(S)$ in the obvious way. However, each of the corresponding isomorphism classes still only contains partitions of a given S. I.e., we've just glommed together the previously defined isomorphism classes of all the different $Par(S)$'s. For this reason, we'll keep $\sim$ and $[B]$ as our notation for this equivalence, whether on a given $Par(S)$ or $Par()$ as a whole. Since the result is the same for $Par(S)$ and the copy of $Par(S)$ in $Par()$, there is no ambiguity in doing so. On the rare occasion when it is necessary to distinguish the two, we'll write $[B]_S$ and $[B']_{S'}$ to make clear that the classes are of $Par()$ and may reside on distinct S's.

As usual, we can take the quotient $Par'() \equiv Par()/\sim$. Clearly, this is just $Par'() = \cup_{S \subseteq U} Par'(S)$, because the quotient is set-by-set. The same holds for $CPar'() \equiv CPar()/\sim$, which is just $CPar'() = \cup_{S \subseteq U} CPar'(S)$.

So far, all we've done is collect our previous isomorphism classes into one big pile. There is no new information or structure in doing so, and it does not embody a notion of cross-S isomorphism classes.

Let's define a new equivalence relation $\sim_a$ on $Par()$ as $B \sim_a B'$ iff (S, B) and (S', B') are isomorphic. I.e., we now allow isomorphisms between partitions of different sets.

> As usual, the dependence on S and S' is implicit in our $\sim_a$ notation.

> $\sim_a$ is an equivalence relation for the same reason that $\sim$ is.

This defines a partition of $Par()$. As always, countable and uncountable partitions can't be isomorphic, so $CPar()$ is compatible with $\sim_a$, and we have a partition of $CPar()$ as well. Denote the classes of this partition — whether of $Par()$ or $CPar()$ — by $[[B]]$.

For illustrative purposes, it is helpful to define another equivalence relation, this time on $Par'()$ and $CPar'()$. Let $[B]_S \sim_s [B']_{S'}$ iff $B \sim_a B'$.

> As we'll see, we can think of $\sim_s$ as transforming $\sim$ into $\sim_a$.

> To see that this is well-defined, suppose $B \sim B_1$ on S and $B' \sim B'_1$ on S' and $B \sim_a B'$. We can pick isomorphisms $f : B \to B_1$ and $g : B' \to B'_1$ and $h : B \to B'$. Then $g \circ h \circ f^{-1}$ is an isomorphism from B_1 to B'_1. This shows that $[B]_S \sim_s [B']_S$ is well-defined, because it does not depend on the choice of representatives. Since $\sim_a$ is an equivalence relation, it readily follows that $\sim_s$ is one too. By the same reasoning as earlier, $CPar'()$ is compatible with $\sim_s$.

We now can define a quotient set in one of two equivalent ways:

- $Par''() \equiv Par'()/\sim_s$ and $CPar''() \equiv CPar'()/\sim_s$
- $Par''() \equiv Par()/\sim_a$ and $CPar''() \equiv CPar()/\sim_a$

I.e., we can view the quotient as happening in one or two stages, with $\sim$ taking us to $Par'()$ as the intermediate step in the first stage. The two-stage view involves distinct sets $Par()$ and $Par'()$ in the two stages, so we can't technically speak of successive 'coarsenings' of partitions — even though this really is what is happening. In the one-stage view, it is clear that $\sim_a$ is a coarsening of $\sim$ on $Par()$ (or $CPar()$).

Clearly, each $[[B]]$ can be expressed as a union of $[]$ classes, one for each S that has at least a single partition isomorphic to B.

We'll denote this $[[B]] = \cup[B_i]$, with the understanding that the union is over such S's, and that the B_i for each S is isomorphic to B. From our previous discussion, it is clear that the particular representatives B and B_i are irrelevant. By a similar token, a given S has at most one $[B_i]$ that is isomorphic to B.

Therefore, $\sim_a$ is a coarsening of $\sim$ on $Par()$ and $CPar()$. This means that the partitions induced by $\sim_a$ on $Par()$ and $CPar()$ are coarsenings of those induced by $\sim$.

Note that, unlike $Par()$ and $Par'()$, Par'' is not graded. I.e., it isn't a union of some $Par''(S)$ over S. There is no meaningful notion of $Par''(S)$ (or, more precisely, $Par''(S)$ would just equal $Par'(S)$).

From proposition 2.22, we know that $(S,B) \sim_a (S',B')$ iff $|S| = |S'|$ and $|B| = |B'|$ and there exists a bijection $k : B \to B'$ s.t. $|b| = |k(b)|$ always. Let's focus on the first part of this for the moment.

Prop 2.41: (i) If $|S| = |S'|$ then $Par(S)$, $CPar(S)$, $Par'(S)$, and $CPar'(S)$ are each bijective with their S' counterpart. (ii) Any given bijection $f : S \to S'$ induces a particular bijection between each of these sets on S and its S' counterpart. (iii) Each class of $[[B]] = \cup[B_i]$ is a union over only S's with the same $|S|$.

Pf: (i,ii) Pick any bijection $f : S \to S'$. Given $B \in Par(S)$, $B' \equiv f'(B)$ is an element of $Par(S')$ and is isomorphic to B, and $(f^{-1})'(B') = f'^{-1}(B') = B$. Therefore, we have established a bijection between $Par(S)$ and $Par(S')$. Formally, this bijection is just $f''|_{Par(S)}$. Since we're dealing with isomorphism, countability respects it, and we thus restrict to a bijection between $CPar(S)$ and $CPar(S')$. Now, suppose that $B \sim B_1$ on S and $B' \sim B_1'$ on S' and $B \sim_a B'$. Since $\sim$ implies $\sim_a$, by the transitivity of $\sim_a$ we have that $B_1 \sim_a B_1'$ as well. This goes the other way too, if we start with $B_1 \sim_a B_1'$ instead of $B \sim_a B'$. We thus see that $B \sim_a B'$ iff $B_1 \sim_a B_1'$. If we are given a specific isomorphism f from B to B', then $B' = f'(B)$, and f establishes a specific bijection between $Par'(S)$ and $Par'(S')$ via $[B]_S \to [f'(B)]_{S'}$. This clearly restricts to a bijection between $CPar'(S)$ and $CPar'(S')$ for the same reason as usual. (iii) is obvious since (S,B) cannot be isomorphic to (S',B') unless $|S| = |S'|$.

What this tells us is that, from the standpoint of partitions of sets, we really only care about $|S|$ rather than S. All S's of the same size look the same to us. They just relabel the elements. Moreover, although $Par''()$ and $CPar''()$ aren't graded by S, they *are* graded by $|S|$. For all practical purposes, $\sim$, $\sim_s$, and $\sim_a$ only care about the cardinality of S. I.e., we lose no generality by restricting ourselves to a single S of each cardinality.

It may be tempting to remove this redundancy and quotient out $Par()$ by the cardinality of S. However, there is no natural way to do this. The problem is that we have no canonical way to associate partitions of S with those of S', even when $|S| = |S'|$. We could pick a specific representative of each cardinality, but this introduces an arbitrary choice and doesn't buy us anything. The redundancy is removed anyway by $[[B]]$, because it lumps together the redundant isomorphism classes of all S's of a given $|S|$. In fact, $[[B]]$ effectively factors this out as part of $\sim_a$. What we do *not* have is a multi-stage way to get to $Par''()$ that involves a quotient by cardinality (as opposed to or in addition to a quotient by $\sim$) as an intermediate step. Fortunately, there is no need for it.

Although it is possible for an infinite set to be bijective with a proper subset, isomorphism is stricter.

Prop 2.42: On a given S, (i) no B can be isomorphic to a proper refinement of itself, (ii) no B can be isomorphic to a proper coarsening of itself, (iii) the singleton partition cannot be isomorphic to any other partition of S, and (iv) the trivial partition cannot be isomorphic to any other partition of S.

Pf: (i) Our earlier results imply this, but let's explicitly show it anyway. It is quite possible for $|B| = |B_R|$ for a proper refinement if $|B|$ is infinite, but an isomorphism is stricter. Suppose an isomorphism $f : S \to S$ exists from B_R to B. Then f is bijective and the induced g and h_b's are bijective. However, for a proper refinement, there must be at least one $b \in B$ s.t. at least two classes b'_R and b''_R have $f(b'_R) \subseteq b$ and $f(b''_R) \subseteq b$. This means that the induced g is non-injective, contradicting our premise. (ii) follows from (i), (iii) follows from (ii), and (iv) follows from (i).

2.10.3. *The set of bijections.* Let's briefly turn our attention to $Bij(S, S')$, the set of bijections between S and S', and consider its properties.

Note that $Bij(S, S')$ is a group iff $S = S'$.

Prop 2.43: Taking the usual liberties with notation, and assuming that S, S', and S'' are nonempty: (i) $Bij(S', S) = Bij(S, S')^{-1}$, (ii) $Bij(S', S'') \circ Bij(S, S') = Bij(S, S'')$, (iii) $Bij(S, S') \neq \emptyset$ iff $|S| = |S'|$, and (iv) $Bij(S, S')$ is bijective with $Bij(S_1, S'_1)$ (and nonempty) iff $|S| = |S_1| = |S'| = |S'_1|$.

Pf: (i) f is a bijection from S to S' iff f^{-1} is a bijection from S' to S. (ii) Bijections compose. (iii) is basic set theory, (iv) Suppose $|S| = |S'| = |S_1| = |S'_1|$. By definition, we can pick a bijection $h : S \to S_1$ and a bijection $k : S' \to S'_1$. We then define a map $\alpha : Bij(S, S') \to Bij(S_1, S'_1)$ via $\alpha(f) \equiv k \circ f \circ h^{-1}$. This clearly is invertible, taking $g \to k^{-1} \circ f \circ h$. For the converse, assume that $Bij(S, S')$ and $Bij(S_1, S'_1)$ are bijective. From (iii), we know that either both are empty or $|S| = |S'|$ and $|S_1| = |S'_1|$. $|Bij(S, S')| = |S'|^S$ and $|Bij(S_1, S'_1)| = |S'_1|^{|S'|}$. These must be equal or we cannot have a bijection between $Bij(S, S')$ and $Bij(S_1, S'_1)$. However x^x is a monotonically increasing function of x. No two distinct cardinals x and y have $x^x = y^y$. For finite values, this is patently true. For infinite x, $\beth_n^{\beth_n} = \beth_{\max(n+1,n)} = \beth_{n+1}$.

Earlier, we discussed the effect of a given bijective function on a given B or a given B'. Let's now ask how the set of all bijective functions looks to B or B'.

Recall from proposition 2.7 that every bijection $f : S \to S'$ defines a bijection between $Par(S)$ and $Par(S')$ via the pairing of isomorphic partners. We saw in proposition 2.6 that the pairing takes the form $B \to f'(B)$ in one direction and $B' \to f^*B'$ in the other. These are inverses, since $f^*B' = (f^{-1})'(B') = f'^{-1}(B')$).

The overall map may be expressed succinctly as $f''|_{Par(S)} : Par(S) \to Par(S')$.

Obviously, each such $(B, f'(B))$ or (f^*B', B') pair resides within the same $[[B]]$ class since $(S, B) \sim_a (S', f'(B))$ and $(S, f^*B') \sim_a (S', B')$. The pairing also respects [] classes within a given $[[B]]$ class, as embodied in the following proposition.

Prop 2.44: Let $f : S \to S'$ be bijective. Then it (i) induces a bijection $\alpha_f : Par(S) \to Par(S')$, (ii) induces a bijection $\alpha'_f : Par'(S) \to Par'(S')$, and (iii) induces a bijection between each $[B] \in Par'(S)$ and the corresponding $\alpha'_f([B]) \in Par'(S')$. (iv) If f_1 and f_2 are distinct bijections from $S \to S'$ then $\alpha_{f_1} \neq \alpha_{f_2}$.

I.e. f pairs each isomorphism class of partitions of S with a unique isomorphism class of partitions of S' and also pairs the elements of each class with those of its partner.

Pf: (i) Via two applications of proposition 1.1 we know that f is bijective iff f' is bijective iff f'' is bijective. Define $\alpha_f \equiv f''|_{Par(S)}$. This takes $B \to f'(B)$. As the restriction of the injective map f', it is injective. Since f is bijective, we saw that $f^*B' = f'^{-1}(B')$, so $f'(f^*B') = B'$. I.e., for each $B' \in Par(S')$, there is some partition f^*B that f' maps to B'. This makes $f''|_{Par(S)}$ surjective (and thus bijective) to $Par(S')$.

Pf: (ii) Let $B_1 \sim B$, and let $k : S \to S$ be some isomorphism from B to B_1. Since f is both an isomorphism from B to $f'(B)$ and an isomorphism from B_1 to $f'(B_1)$, $f \circ k \circ f^{-1}$ is an isomorphism from $f'(B)$ to $f'(B_1)$. We therefore have $f'(B) \sim f'(B_1)$. The converse holds too, swapping f and f^{-1}. So $B \sim B_1$ iff $f'(B) \sim f'(B_1)$. We thus have a bijection $\alpha'_f : Par'(S) \to Par'(S')$, defined by $\alpha'_f([B]) = [f'(B)]$.

Pf: (iii) Since we have a pairing of isomorphism classes, we know that for any given $[B]$, the restriction $f''|_{[B]} : [B] \to [f'(B)]$ is bijective (with $[B]$ and $[f'(B)]$ regarded as sets of partitions rather than elements of $Par'(S)$ and $Par'(S')$).

Pf: (iv) Since $f_1 \neq f_2$, there exists some $x \in S$ s.t. $f_1(x) = y_1$ and $f_2(x) = y_2$. Consider $B = \{(x), S - \{x\}\}$. The isomorphic partners assigned by f_1 and f_2 are $\{(y_1), S' - \{y_1\}\}$ and $\{(y_2), S' - \{y_2\}\}$, which differ. So the induced α_{f_1} and α_{f_2} differ.

This result is a bit deceptive. For each bijection $f : S \to S'$, proposition 2.44 establishes three types of bijections: (i) $Par(S) \to Par(S')$, (ii) $Par'(S) \to Par'(S')$, and (iii) between the partitions within each $[B]$ and those within its partner $[f'(B)]$ (regarded as sets of partitions). Although (i) and (iii) are f-specific, (ii) is not.

Prop 2.45: Given S and S' s.t. $|S| = |S'|$: (i) there exists a natural bijection $\eta : Par'(S) \to Par'(S')$ and (ii) for every $f \in Bij(S, S')$, $\eta = \alpha'_f$.

I.e., $Bij(S, S')$ respects isomorphism classes, but in a trivial way. If we ask what set of maps from S to S' take a given isomorphism class to its partner, the answer is 'every bijective map'. The only difference is how each such f matches the partitions within a class to those of its partner class.

Pf: Consider some $B \in Par(S)$. Suppose that f_1 and f_2 induce distinct maps α'_{f_1} and α'_{f_2}. Then f_1 is an isomorphism from B to some B'_1 and f_2 is an isomorphism from B to some B'_2. However, this means that $f_2 \circ f_1^{-1} : S' \to S'$ is an isomorphism from B'_1 to B'_2. Therefore $[B'_1] = [B'_2]$. The same argument holds going the other way. Therefore $\alpha'_{f_1} = \alpha'_{f_2}$.

We already know that if $|S| = |S'|$, each $[B]$ has a unique partner $[B']$. It therefore should come as no surprise that each f induces this partnering, even if it wasn't immediately evident from our proof of proposition 2.44.

Although every bijection induces the same α'_f, each induces a different α_f. In fact, distinct bijections must induce α_f's which differ on at least two partitions.

Why two and not one? If α_{f_1} partners B with B' and α_{f_2} partners B with B'_1, then α_{f_2} must partner some other B_1 with B'. The smallest difference we can have is to swap one pair of partners.

Prop 2.46: Given S and S' s.t. $|S| = |S'| > 2$, every $f \in Bij(S, S')$ induces a distinct α_f.

I.e., we have a bijection between $Bij(S, S')$ and the set of induced α_f's. Even though distinct α_f's can map a given B to the same B', they will differ in how they map *some* partition.

Pf: Suppose f_1 and f_2 are bijections and $f_1(x) = y$ and $f_2(x) = y'$ (with $y \neq y'$). Let $B = \{(x), (S - \{x\})\}$ (i.e. a 2-class partition, with a singlet as one of the classes). Then $\alpha_{f_1}(B) = \{(y), (S - \{y\})\}$ and $\alpha_{f_2}(B) = \{(y'), (S - \{y'\})\}$. Since $|S| > 2$, these two partitions cannot be the same. Therefore $\alpha_{f_1} \neq \alpha_{f_2}$, because they differ on B.

If $|S| = |S'| = 2$, there are two bijections $S \to S'$ and $|Par(S)| = |Par(S')| = 2$. The map α_f is unique since there is no choice in how to pick partners (the singleton partitions are isomorphic and the trivial partitions are isomorphic). By contrast, in the case of $|S| = |S'| = 3$, there are 6 bijections and $Par(S) = Par(S') = 5$. As always, the singleton partitions must partner and the trivial partitions must partner. The only freedom is in partnering the 3 partitions on each end that involve a $2 - 1$ split of elements. There are 6 possible pairing functions, and these correspond to our 6 bijections. Now, suppose $|S| = |S'| = n$ for some finite n. There are $n!$ bijections between S and S'. A given α_f can only partner a partition with another partition that has matching class sizes. Let's call the number of classes and their sizes (i.e. $|B|$ and $\{|b_i|\}$) a "profile" for a partition. Each α_f is profile-preserving. If there are m distinct partitions of a given profile on S, then there are $m!$ possible ways to partner them with the m partitions of that profile on S'. If the product over profiles $\prod m_i! < n!$, then (by the pigeonhole principle) some two f's must induce the same α_f. Our proposition tells us that this can't happen, but let's see how the combinatorics cooperate. Define an 'almost-singleton' partition to look like the singleton partition but with two singletons merged. Ex. $\{(1), (2), (3), (4, 5)\}$. There are $n(n-1)/2$ ways to construct such a partition. So, we have $m = n(n-1)/2$, which is $\geq n$ for $n > 2$. Note that we've ignored all the other profiles, and these contribute too. In fact, we quickly end up with a vastly higher number of bijections between $Par(S)$ and $Par(S')$ than between S and S'. This illustrates that not only isn't the combinatorics an obstruction, but many of the bijections between $Par(S)$ and $Par(S')$ are not induced by any $f \in Bij(S, S')$.

2.11. **A quotient lattice?** Let's return to a single S for a moment. We've established a notion of isomorphism classes [] under $\sim$. Specifically, $(S, B_1) \sim (S, B_2)$ iff B_1 and B_2 are isomorphic. Our lattice of partitions of S involves a partial order, a meet, a join, and maximum and minimum elements. Let's see if we can construct a corresponding quotient lattice that embodies these same concepts. Unless otherwise stated, all partitions and isomorphism classes are on the same S.

In our partition lattice, $B \leq B'$ iff B is a refinement (proper or not) of B. $B \leq B'$ and $B' \leq B$ iff $B = B'$. We can define $B < B'$ iff $B \leq B'$ and $B \neq B'$. This converts our weak partial order to a strict one, with 'proper refinement' as the embodied concept.

We saw in proposition 2.42 that if $B \sim B'$, then $B \not< B'$ and $B' \not< B$, since a proper coarsening or proper refinement of B cannot be isomorphic to it.

Let's define $[B] < [B']$ iff for every $B \in [B]$, $B < B'$ for some $B' \in [B']$. At this point, $<$ is just a symbol. We haven't shown it to be an order relation. This definition may seem needlessly strict, but the following proposition shows that it is equivalent to the obvious lax alternative.

Prop 2.47: The following conditions are equivalent:

- (i) $[B] < [B']$.
- (ii) For every $B' \in [B']$, $B < B'$ for some $B \in [B]$.
- (iii) For some $B \in [B]$ and some $B' \in [B']$, $B < B'$.
- (iv) At least one pair of elements drawn from $[B]$ and $[B']$ is comparable, and for every comparable pair $B \in [B]$ and $B' \in [B']$, we have $B < B'$.

Note that most pairs are not comparable.

Pf: (i$\rightarrow$ ii) Suppose that $[B] < [B']$ but there is some $B'_1 \in [B']$ s.t. there is no $B_1 \in [B]$ for which $B_1 < B'_1$. Pick some $B \in [B]$. Since $[B] < [B']$, there is some $B' \in [B']$ s.t. $B < B'$. Let f be some isomorphism from B' to B'_1. I.e. we pick an f s.t. $B'_1 = f'(B')$. Since all the partitions are on the same S, f is also an isomorphism from B to $B_1 \equiv f'(B)$. Consider any $b_1 \in B_1$. $f^{-1}(b_1)$ is some $b \in B$. Since $B < B'$, $b \subseteq b'$ for some $b' \in B'$. Therefore $f(b) \subseteq f(b')$. However, $f(b) = f(f^{-1}(b_1)) = b_1$, and $f(b')$ is some $b'_1 \in B'_1$. So $b_1 \subseteq b'_1$, and $B_1 < B'_1$.

Pf: (ii$\rightarrow$ i) Let (ii) hold. Consider some $B \in [B]$, and pick some $B'_1 \in [B']$. By (ii), there is some $B_1 \in [B]$ s.t. $B_1 < B'_1$. Since B and B_1 are isomorphic, pick isomorphism f from B_1 to B. Since all partitions are on the same S, f is also an isomorphism from B'_1 to $B' \equiv f'(B'_1)$. Consider $b \in B$. $f^{-1}(b) = b_1$ for some $b_1 \in B_1$, and $b_1 \subseteq b'_1$ for some $b'_1 \in B'_1$. Therefore $f(b_1) \subseteq f(b'_1)$. The left side is $f(f^{-1}(b)) = b$, and the right side is some $b' \in B'$. Therefore, $B < B'$. Since this holds for all $B \in [B]$, we have $[B] < [B']$.

Pf: (i$\rightarrow$ iii) This is automatic since if it holds for all, it holds for at least one.

Pf: (iii$\rightarrow$ i) Suppose (iii) holds, and let $B < B'$ be the relevant pair. Pick some other $B_1 \in [B]$, and let $f : B \rightarrow B_1$ be an isomorphism. For a given $b_1 \in B_1$, $f^{-1}(b_1) \in B$. Let's call it b. Since $B < B'$, $b \subseteq b'$ for some $b' \in B'$, and $f(b) \subseteq f(b')$. Since all partitions are on the same set, f is also an isomorphism from B' to $B'_1 \equiv f'(B')$. Since $f(b) = f(f^{-1}(b_1)) = b_1$ and $f(b')$ is some $b'_1 \in B'_1$, we have $b_1 \subseteq b'_1$. This means $B_1 < B'_1$. Since every B has a partner, $[B] < [B']$.

Pf: (i$\rightarrow$ iv): That there exists a pair is automatic from the definition of $[B] < [B']$, so we need only show that every comparable pair has the correct $<$. Let $B < B'$ be the relevant pair. Suppose that $B'_1 < B_1$ for some $B_1 \in [B]$ and $B'_1 \in [B']$. By (iii), we have both $[B] < [B']$ and $[B'] < [B]$. Since $[B'] < [B]$, (ii) tells us that B has some partner $B'_2 \in [B']$ s.t. $B'_2 < B$. Therefore $B'_2 < B < B'$, which means that B'_2 is a proper refinement of B'. However, a partition cannot be isomorphic to a proper refinement. Therefore, we can't have $B'_1 < B_1$. Either B_1 and B'_1 are not comparable or $B_1 < B'_1$.

Pf: (iv$\rightarrow$ i): Suppose $B < B'$ for some pair. Then (iii) gives us (i).

Prop 2.48: $[B] < [B']$ is antisymmetric and transitive.

Pf: (antisymmetric): Suppose $[B] < [B']$ and $[B'] < [B]$. Pick $B \in [B]$. Since $[B] < [B']$, $B < B'$ for some $B' \in [B']$. But $[B'] < [B]$ too, so $B' < B_1$ for some $B_1 \in [B]$. On $Par(S)$, $<$ is an order relation (and thus transitive), so $B < B' < B_1$ implies $B < B_1$. However, $B \sim B_1$ and by proposition 2.42, B can't be isomorphic to a proper coarsening. (transitive): Suppose $[B] < [B']$ and $[B'] < [B'']$. Pick B. Then $B < B'$ for some B' and $B' < B''$ for some B''. Again by the transitivity of $<$ on $Par(S)$, we get $B < B''$. Therefore, every B has a B'' s.t. $B < B''$. This is the definition of $[B] < [B'']$.

We therefore have a strict partial order on $Par'(S)$.

We can make this into a weak partial order easily enough, by defining $[B] \leq [B']$ if $[B] < [B']$ or $[B] = [B']$.

It immediately follows that the singleton class is $<$ every other class and every other class is $<$ the trivial class. Those two classes therefore are minimal and maximal elements of the poset $Par'(S)$.

We know from proposition 2.42 that the isomorphism class of the singleton partition consists of it alone and the isomorphism class of the trivial partition consists of it alone. Therefore, our maximum and minimum elements are single-partition isomorphism classes.

We thus see that $Par'(S)$ is a poset with both maximal and minimal elements. We're halfway to a lattice, but, sadly, this as far as we can go. The problem is that we cannot define meaningful join or meet operations between classes. Let's see what fails.

If we naively try to define $[B] * [B'] \equiv [B * B']$, we run into an immediate problem: it is not well-defined. If $B, B_1 \in [B]$ and $B', B'_1 \in [B']$, it doesn't follow that $B * B'$ and $B * B'_1$ and $B_1 * B'$ and $B_1 * B'_1$ are all in the same isomorphism class.

Ex. let $S = \{1, 2, 3, 4\}$ and $B = B' = \{(1, 2), (3, 4)\}$ and $B_1 = B'_1 = \{(1, 3), (2, 4)\}$. All of these are isomorphic with one another. $B * B' = B = B'$ and $B_1 * B'_1 = B_1 = B'_1$ are in the same isomorphism class, but $B_1 * B' = B * B'_1 = \{(1), (2), (3), (4)\}$ is not.

We could try something like $[F([B]) * F([B'])] = [F([B] \cup [B'])]$, where $F([B])$ here denotes our earlier $F(Z)$ with Z is just $[B]$ viewed as a set of partitions. I.e., we take the isomorphism class of the common refinement of all the partitions in $[B]$ and $[B']$. However, it is easy to see that this is just the singleton partition. In fact, any individual $F([B])$ is just the singleton partition.

Take any partition B of S and construct a new partition B_1 by swapping a single element x of class b_1 with a single element y of class b_2. Call the new classes b'_1 and b'_2. Then B and B_1 clearly are isomorphic. $B * B_1$ consists of (i) all classes of B except b_1 and b_2, (ii) $b_1 \cap b'_1 = (b_1 - \{x\})$ and $b_2 \cap b'_2 = (b_2 - \{y\})$, and (iii) $b_1 \cap b'_2 = \{x\}$ and $b_2 \cap b'_1 = \{y\}$. Since the singletons in (iii) can't get any smaller, they persist in all subsequent $*$ operations with other partitions. Since we can do this with any x and y, it follows that the common refinement of $[B]$ has to be the singleton partition.

This problem cannot be remedied, at least in any fashion that produces a conceptually meaningful join operation between classes. The same sort of obstacle arises in attempting to define a meet operation for classes. We simply cannot construct a quotient lattice.

In summary, $Par(S)$ is a complete lattice, but $Par'(S)$ is merely a quotient poset with maximal and minimal elements.

2.12. **Morphism Sets.** As before, denote by $Map(S, S')$ the set of all maps from S to S' and by $Surj(S, S')$ the set of all surjective maps. Obviously, $Bij(S, S') \subset Surj(S, S') \subset Map(S, S')$.

Just like $Iso(B, B')$, we can define the following for a given (S, B) and (S', B'):

- $Mor(B, B')$: The morphisms (aka 'flexible maps into') from B to B'.
- $SMor(B, B')$: The surjective morphisms (aka 'maps into') from B to B'.
- $CMor(B, B')$: The coarsening morphisms from B to B', including isomorphisms.
- $UMor(B, B')$: The uncoarsening nonmorphisms from B to B', including isomorphisms.
- $NCMor(B, B')$: The nonisomorphism coarsening morphisms from B to B' (i.e. excluding isomorphisms).
- $NUMor(B, B')$: The nonisomorphism uncoarsening nonmorphisms from B to B', (i.e. excluding isomorphisms).
- $MapTo(B, B')$: The maps 'to' B' from B.
- $FMapTo(B, B')$: The flexible maps 'to' B' from B.

We have the following relationships:

We'll write $[A, B] \subset C$ as a concise way of saying that both $A \subset C$ and $B \subset C$.

- $Iso(B, B') \subseteq [CMor(B, B'), MapTo(B, B')] \subset SMor(B, B') \subset Mor(B, B') \subset Map(S, S')$

  $UMor(B, B')$ is not in this chain because a nonisomorphism uncoarsening nonmorphism can't be a morphism from B to B'.

  Note that $CMor(B, B')$ and $MapTo(B, B')$ represent two distinct generalizations of isomorphisms. $CMor(B, B')$ allows the induced g to be noninjective, and $MapTo$ allows the induced h_b's to be noninjective. If we allow both to be noninjective, we get an ordinary surjective morphism — but not the most general surjective morphism, which also allows cross-b noninjectivity for b's that map to the same b'. As we'll see momentarily, $NCMor$ cannot coexist with $Iso(B, B')$. $CMor$ either equals $Iso(B, B')$ or it equals $NCMor(B, B')$, and the other must be empty. On the other hand, $MapTo$ can coexist with $Iso(B, B')$, augmenting it with noninjective f's that still produce bijective induced g's.
- $Iso(B, B') \subseteq MapTo(B, B') \subset FMapTo(B, B') \subset Mor(B, B') \subset Map(S, S')$

  Note that $MapTo(B, B')$ appears in both of the above chains. The notion of 'map to' generalizes in distinct directions to 'surjective morphism' and 'flexible map to'.
- $NCMor(B, B') \subset CMor(B, B') \subset SMor(B, B') \subset Mor(B, B') \subset Map(S, S')$
- $NUMor(B, B') \subset UMor(B, B')$
- $[SMor(B, B'), MapTo(B, B')] \subset Surj(B, B')$
- $[Iso(B, B'), CMor(B, B'), UMor(B, B'), NCMor(B, B'), NUMor(B, B')] \subset Bij(S, S')$
- $Iso(B, B') = UMor(B, B') \cap CMor(B, B')$
- $NCMor(B, B') \cap NUMor(B, B') = \emptyset$
- $UMor(B, B') = CMor(B', B)^{-1}$
- $NUMor(B, B') = NCMor(B', B)^{-1}$

The inclusion ones are obvious from the definitions. We'll prove the others shortly.

It also is clear that not all of these can simultaneously be nonempty, as the following proposition illustrates:

Prop 2.49: Suppose that $|S| = |S'|$. (i) If B and B' are isomorphic, then $Iso(B, B') \subseteq MapTo(B, B')$ and $CMor(B, B') = UMor(B, B') = Iso(B, B')$ and $NCMor(B, B') = NUMor(B, B') = \emptyset$. (ii) If B and B' are nonisomorphic, then $Iso(B, B') = \emptyset$, $CMor(B, B') = NCMor(B, B')$, $UMor(B, B') = NUMor(B, B')$, and at most one of $CMor(B, B')$ and $UMor(B, B')$ is nonempty.

This tells us that $NCMor$ can't coexist with Iso, and our designation of "nonisomorphism coarsening morphism" really distinguishes nonconcomitant cases. It's not that $CMor$ can contain isomorphisms and nonisomorphism coarsening morphisms. Any given $CMor(B, B')$ consists of *just* isomorphisms or of *just* nonisomorphism coarsening morphisms (or of nothing at all).

We shouldn't confuse the last part of (ii) with our earlier result that $UMor(B', B) = CMor(B, B')^{-1}$. The latter must either both be empty or both be nonempty. However, in the current proposition we're talking about $UMor(B, B')$ vs $CMor(B, B')$, not $UMor(B', B)$ vs $CMor(B, B')$.

The last part of (ii) may seem counterintuitive. Suppose $|B| = |B'|$ and $|S| = |S'|$ but B and B' are not isomorphic. Then it may seem like we could simultaneously have coarsening morphisms in both directions. Ex, if $|B| = |B'| = \aleph_0$ and each class of B or B' has size $\aleph_0$, we could merge classes of B into B' and classes of B' into B. However, two issues arise: (a) any such situation also requires that there exist a bijective way to match up equal-sized classes of B and B' and (b) since coarsening morphisms are bijections, they would have to be isomorphisms in this case anyway. In fact, we saw this explicitly in proposition 2.29.

Pf: (i) We already know that $Iso(B, B') \subseteq MapTo(B, B')$. Suppose that B and B' are isomorphic, and let f be a nonisomorphism coarsening morphism from B to B'. Then B is isomorphic to some proper refinement B'_R of B'. Since being isomorphic is a transitive property, this means that B' is isomorphic to a proper refinement of itself, which proposition 2.42 tells us is impossible. $NUMor$ and $NCMor$ therefore must be empty, and $CMor$ and $UMor$ must equal Iso. (ii) If B is not isomorphic to B' then $Iso(B, B')$ has to be empty, which means that $NCMor = CMor$ and $NUMor = UMor$. If bijections between S and S' exist, the latter sets may (but need not be) nonempty. Suppose that both $CMor(B, B')$ and $UMor(B, B')$ are nonempty. Then B is isomorphic to some proper B'_R and B is isomorphic to some proper B'_C, which means that B'_R is isomorphic to B'_C, an impossibility. Therefore, at most one of $CMor(B, B')$ and $UMor(B, B')$ can be nonempty. It is also possible for both to be empty, as the following example illustrates.

Ex. where $|S| = |S'|$ and neither B nor B' is a coarsening of the other (and they're not isomorphic). Let $S = S' = \{1, 2, 3, 4\}$, and let $B = \{(1, 2), (3, 4)\}$ and $B' = \{(1), (2, 3, 4)\}$. There is no isomorphism, coarsening morphism, uncoarsening nonmorphism or map 'to' between them. However, there are flexible maps 'to' (ex. $f : (1, 2, 3, 4) \to (1, 1, 2, 2)$).

For convenience, we'll sometimes want to make statements that apply to several or all of $Iso(B, B')$, $Mor(B, B')$, $SMor(B, B')$, $CMor(B, B')$, $UMor(B, B')$, $NCMor(B, B')$, $NUMor(B, B')$, $MapTo(B, B')$, and $FMapTo(B, B')$. We'll refer to these collectively as our nine $FOO(B, B')$ sets. When $FOO(B, B')$ appears, it can be any of these. An expression like $FOO(B', B'') \circ FOO(B, B')$ means that the same type appears in both places (ex. $Mor(B', B'') \circ Mor(B, B')$, but not $Iso(B', B'') \circ Mor(B, B')$).

When a statement only applies to a few of these, we'll enumerate them explicitly. Ex. blah blah holds for $FOO(B, B')$ with FOO any of Iso, $CMor$, or $UMor$.

We'll also sometimes just write Mor or FOO in place of $Mor(B, B')$ or $FOO(B, B')$ when no ambiguity can arise.

Prop 2.50: The cardinalities of S, S', B, and B' constrain our sets of maps in the following ways:

- (i) If $|S| > |S'|$ then $CMor$, $UMor$, $NCMor$, $NUMor$, and Iso are $\emptyset$.
- (ii) If $|S| < |S'|$, then all but Mor and $FMapTo$ are $\emptyset$.
- (iii) If $|B| < |B'|$ then all but $UMor$ and $NUMor$ are $\emptyset$.
- (iv) If $|B| > |B'|$ then Iso, $MapTo$, $FMapTo$, $UMor$, and $NUMor$ are $\emptyset$.
- (v) If $|S| = |S'|$ and $|B| = |B'|$ but there exists no bijective $f : S \to S'$ s.t. $f'|_B$ maps classes of the same cardinality, then Iso is $\emptyset$.

Note that it is quite possible to have a nonisomorphism coarsening morphism with $|B| = |B'|$. Ex., on the integers, Id_N is a nonisomorphism coarsening morphism from the singlet partition to $\{(1, 2), (3, 4), (5, 6), \dots\}$, even though $|S| = |S'|$ and $|B| = |B'|$.

Pf: (i,ii) $CMor$, $UMor$ $NCMor$, $NUMor$, and Iso consist of bijections, and there can exist no bijection if $|S| \neq |S'|$. $SMor$ and $MapTo$ consist of surjections, and there can exist no surjection from S to S' if $|S| < |S'|$. (iii) All morphisms require a surjective induced g. This is impossible if $|B| < |B'|$. (iv) Isomorphisms and maps 'to' and flexible maps 'to' all require a bijective induced g. An uncoarsening nonmorphism requires that f^{-1} induces a surjective g, which is impossible if $|B| > |B'|$. (v) An isomorphism requires that each h_b be a bijection, and this is impossible unless we can ensure that every $|b| = |g(b)|$.

The following result tells us that there is no point in defining things like $Mor()$ or $Mor(-,B')$ or $Mor(B,-)$, etc, because these include every relevant map or are otherwise uninteresting.

Prop 2.51: Given some B and B':

- (i) $Mor() = Mor(-,B') = Mor(B,-) = Map(S,S')$
- (ii) $SMor() = SMor(-,B') = SMor(B,-) = Surj(S,S')$
- (iii) $CMor() = CMor(-,B') = CMor(B,-) = Bij(S,S')$
- (iv) $UMor() = UMor(-,B') = UMor(B,-) = Bij(S,S')$
- (v) If $|S| > 1$, then $NCMor() = NCMor(-,B') = NCMor(B,-) = Bij(S,S')$
- (vi) If $|S| > 1$, then $NUMor() = NUMor(-,B') = NUMor(B,-) = Bij(S,S')$
- (vii) $Iso() = Iso(-,B') = Iso(B,-) = Bij(S,S')$
- (viii) $MapTo(-,B') = MapTo(-,-) = Surj(S,S')$. However, $MapTo(B,-) \subset Surj(S,S')$.
- (ix) $FMapTo(-,-) = Map(S,S')$. However, $FMapTo(B,-) \subset Map(S,S')$ and $FMapTo(-,B') \subset Map(S,S')$.

Pf: (i-ii) Since a surjective morphism is the same as a 'map into' and a morphism is a 'flexible map into', (ii) just restates parts of proposition 1.2, and the addendum to that section confirms that (i) holds as well.

Pf: (iii-iv) follow from proposition 2.17.

Pf: (v-vi) is easy to see as well. As with (iii-iv), we get this from proposition 2.17. A singleton partition can only be isomorphic to a singleton partition and a trivial partition can only be isomorphic to a trivial partition. If we require that $|S| > 1$, then $|S'| > 1$ (or $Bij(S,S') = \emptyset$), and the singleton and trivial partitions are distinct. In that case, there is a coarsening morphism from some non-trivial partition to the trivial partition and a coarsening morphism from the singleton partition to some non-trivial partition and an uncoarsening nonmorphism from the trivial partition to some non-trivial partition and an uncoarsening nonmorphism to the singleton partition from some non-singleton partition. None of these can be isomorphisms, so we have shown that every bijective f appears in $NCMor(B,-)$ and $NCMor(-,B')$ and thus $NCMor(-,-)$, and ditto for its inverse and $NUMor$ with the roles of B and B' swapped.

Pf: (vii) follows from proposition 2.6.

Pf: (viii) follows from proposition 1.2, which tells us that every $f \in Surj(S,S')$ maps each B 'to' at most one B' and maps some unique B 'to' each B'. I.e., every surjective f appears in $MapTo(-,B')$ and thus $MapTo(-,-)$. However, $MapTo(B,-)$ contains only those surjective f's whose noninjectivity is solely intraclass.

Pf: (ix) follows from the modified proposition 1.2 in the addendum to that section. There, we saw that each f flexibly maps B 'to' at most one B' and flexibly maps 'to' B' from at most one B. More specifically, B has a partner iff f only has intraclass noninjectivity, and B' has a partner iff no b' is entirely outside the image of f (i.e. $f^{-1}(b') \neq \emptyset$). As for $FMapTo(-,-)$, consider some $f \in Map(S,S')$. Define a class b' of B' for each $x' \in \text{Im } f$, which doesn't give us a partition if f is not surjective. To remedy this, allocate any unmapped-to space in any way we wish between the classes of B'. Let $B = f^*B'$. Then B is a partition of S and B' is a partition of S', and the induced g is bijective by construction. We thus have a flexible map from B to B'. I.e., every f is a flexible map 'to' some partition from some partition.

It follows that things like $FOO(B,-)$ and $FOO(-,B')$ and $FOO(-,-)$ are uninteresting, aside (potentially) from $MapTo(B,-)$, $FMapTo(B,-)$, and $FMapTo(-,B')$.

As it happens, we won't be particularly interested in these three. However, they remain the only ones which are nontrivial.

Prop 2.52: The following composition and inversion rules hold:

- (i) $FOO(B',B'') \circ FOO(B,B') \subseteq FOO(B,B'')$. Equality is guaranteed only for Iso and if $Iso(B,B') \neq \emptyset$ and $Iso(B',B'') \neq \emptyset$.
- (ii) $Aut(B') \circ FOO(B,B') = FOO(B,B') \circ Aut(B) = FOO(B,B')$
- (iii) $PAut(B') \circ FOO(B,B') = FOO(B,B') \circ PAut(B) = FOO(B,B')$

- (iv) $Iso(B,B')^{-1} = Iso(B',B)$
- (v) $CMor(B,B')^{-1} = UMor(B',B)$
- (vi) $NCMor(B,B')^{-1} = NUMor(B',B)$

In (i), we need the caveat to avoid the case where B and B'' are isomorphic, but B' isn't isomorphic to them.

Pf: (i) $\subseteq$ is immediate from proposition 2.21. Equality fails in most cases, because B' may force the loss of information.

Pf: (i) equality for Iso: Let $f \in Iso(B,B'')$. f is bijective, so f^{-1} is a function. Since we assumed that $Iso(B',B'')$ is nonempty, pick some $f_2 \in Iso(B',B'')$, and consider $f_1 \equiv f_2^{-1} \circ f : B \to B'$. As the composition of isomorphisms it is an isomorphism, and we're done.

(i) Counterexample to equality for Mor: Let $S = S'' = \{1,2,3,4\}$ and $S' = \{1,2\}$. Let B and B' be the singleton partitions of S and S', and let $B'' = \{(1,2),(3,4)\}$. Consider $f = (1,2,3,4) \to (1,2,3,4)$. This is a morphism from B to B''. However, there are no morphisms $f_2 \circ f_1 = f$, because any morphism f_2 from B' to B'' leaves empty space.

(i) Counterexample to equality for $SMor$: Let $S = S' = \{1,2,3,4,5,6\}$ and $S'' = \{1,2\}$. Let B and B'' be the singleton partitions and let $B' = \{(1,2,3,4),(5,6)\}$. Any surjective morphism from B to B'' will split the 6 singlets between (1) and (2). However, the only surjective morphisms that can come through B' must result in a $2-4$. For example, $f : (1,2,3,4,5,6) \to (1,1,1,2,2,2)$ has no decomposition into $f_2 \circ f_1$.

(i) Counterexample to equality for $MapTo$: Let $S = \{1 \dots 12\}$, $S' = \{1 \dots 7\}$, and $S' = \{1,2,3,4\}$. Let B, B', and B'' each consist of 2 classes, with $|b_1| = 4, |b_2| = 8$, $|b'_1| = 2$, $|b'_2| = 5$, $|b''_1| = 3$, and $|b''_2| = 1$. For a surjective morphism (including a 'map to'), each class can only map to a class of smaller or equal size. This means that any f_1 must take b_1 to b'_1 and b_2 to b'_2. This is the only permissible g_1. Similarly, the only permissible g_2 takes b'_1 to b''_2 and b'_2 to b''_1. We therefore have no freedom in the choice of g_1 and g_2. Their composition must take b_1 to b''_2 and b_2 to b''_1. However, f itself is not thus constrained. Pick any f that takes b_1 to b''_1 and b_2 to b''_2 surjectively and we have an example of a 'map to' which cannot be decomposed into 'maps to', even allowing for one of those two maps to be an isomorphism.

(i) Counterexample to equality for $FMapTo$: We have more (no pun intended) flexibility with 'flexible maps to' than 'maps to', but we still aren't guaranteed equality. Yes, we needn't be surjective to each class and no longer require that classes map to classes of smaller or equal size. However, this doesn't mean we have carte blanche. Given an f, we still must be able to funnel all of its information through some g_1 and g_2. A simple example suffices for this. Let $S = S'' = \{1,2,3\}$, $S' = \{1\}$, and all three B's be the trivial partitions. Consider $f : (1,2,3) \to (1,2,2)$, a deliberately nonsurjective flexible map 'to'. There exists a flexible map 'to' B' from B (i.e. $f_1 : (1,2,3) \to (1,1,1)$), and a flexible map 'to' B'' from B' (ex. $f_2 : (1) \to (1)$), so $FMapTo(B,B')$ and $FMapTo(B',B'')$ are nonempty. However, there clearly is no way to funnel f through B' because the only map from B to B' loses all the information.

(i) Counterexample to equality for $NCMor$, $NUmor$, $CMor$, and $UMor$: Let $S = S' = S'' = \{1 \dots 16\}$, and let B consist of 4 class of sizes $|b_1| = 2$, $|b_2| = 3$, $|b_3| = 4$, and $|b_4| = 7$, let B' consist of 3 classes of sizes $|b'_1| = 4$, $|b'_2| = 5$, and $|b'_3| = 7$, and let B'' consist of 2 classes of sizes $|b''_1| = 9$ and $|b''_2| = 7$. There clearly are coarsening morphisms from B to B'. Ex. any f_1 that merges b_1, b_2, and b_3 into b'_2 and takes b_4 to b'_3. In fact, it is easy to see that *any* coarsening morphism from B to B' must do so, simply based on size constraints. Similarly, there are coarsening morphisms from B' to B''. Ex. any f_2 that merges b'_1 and b'_2 into b''_1 and takes b'_3 to b''_2. Once again, it is easy to see that *any* coarsening morphism from B' to B'' must do so. Therefore, $CMor(B,B')$ and $CMor(B',B'')$ are not empty. Consider any coarsening morphism f from B to B'' that merges b_1 and b_4 into b''_1 and merges b_2 and b_3 into b''_2. We've just seen that all coarsening morphisms from B to B' must take b_4 to b'_3 and all coarsening morphisms from B' to B'' must take b'_3 to b''_2. There is no way to find two coarsening morphisms whose composition merges b_4 with b_1 to get b''_1, let alone a pair of the form f_1 and $f_2 \equiv f \circ f_1^{-1}$ which does so. Therefore, there is no suitable decomposition. Nor does allowing f_1 or f_2 to be an isomorphism help, because none of the three partitions are isomorphic to one another (they have different numbers of classes). We thus fail to have equality for $NCMor$ and $CMor$. Since uncoarsening nonmorphisms are the inverses of coarsening morphisms, this implies a failure of equality for $NUMor$ and $UMor$ as well. If equality held for $NUMor$, then $NUMor(B',B) \circ NUMor(B'',B') = NUMor(B'',B)$. Taking the proofs below of (v) and (vi) as given, $NUMor(B',B) = NCMor(B,B')^{-1}$, $NUMor(B'',B') = NCMor(B,B'')^{-1}$, and $NUMor(B'',B) = NCMor(B,B'')^{-1}$, and all three are nonempty. Let f be a non-decomposable coarsening morphism from B to B'' as described above. Then $f^{-1} \in NUMor(B'',B)$. If equality held for $NUMor$, then there exists $l_1 \in NUMor(B',B)$ and $l_2 \in NUMor(B'',B')$ s.t. $f^{-1} = l_1 \circ l_2$. However, this would mean that $l_1^{-1} \in NCMor(B,B')$ and $l_2^{-1} \in NCMor(B',B'')$ and $(l_1 \circ l_2) = f^{-1}$. Taking the inverse, $f = (l_1 \circ l_2)^{-1} = l_2^{-1} \circ l_1^{-1}$. We therefore have a decomposition of f into $f_2 \circ f_1$. This guarantees equality for $NCMor$ as well, which we already saw is not the case. Equality exists for $NUMor$ iff it holds for $NCMor$, and we saw that it does not hold for $NCMor$.

Pf: (ii) $\subseteq$ follows directly from proposition 2.21 part viii, which says that the composition of an isomorphism with any map of type FOO is a map of the same type FOO. On the other hand, since $FOO(B,B') \circ Id_S = Id_{S'} \circ FOO(B,B') = FOO(B,B')$, $FOO(B,B') \subseteq FOO(B,B') \circ Aut(B)$ and $FOO(B,B') \subseteq Aut(B') \circ FOO(B,B')$, so we have equality.

Pf: (iii) The same argument as for (ii) holds here. $FOO(B,B') \circ PAut(B) \subseteq FOO(B,B') \circ Aut(B) = FOO(B,B')$, and the same on the other side. On the other hand, $Id_S \in PAut(B)$ and $Id_{S'} \in PAut(B')$, so we still can use these to show that $FOO(B,B') \subseteq FOO(B,B') \circ PAut(B)$, etc.

Pf: (iv) just restates proposition 2.39, part i.

Pf: (v) just restates proposition 2.15.

Pf: (vi) follows from (iv) and (v). If f is an isomorphism, then f^{-1} is an isomorphism, so if we disallow isomorphisms, both f and f^{-1} must be nonisomorphisms.

We saw earlier that any two isomorphisms from B to B' are related by an element of $Aut(B)$ (or, equivalently, by an element of $Aut(B')$). What about two morphisms?

In proposition 2.52, part ii, we saw that if f is of type $FOO(B, B')$ then any $k \circ f \circ h$ (with $k \in Aut(B')$ and $h \in Aut(B)$) is of type $FOO(B, B')$. However, this does *not* tell us that (i) every f_1 and f_2 of type $FOO(B, B')$ are related via $f_2 = k \circ f_1 \circ h$ for some k and h or (ii) that $Aut(B') \circ f = f \circ Aut(B)$ for a given f or (iii) that $FOO(B, B') = Aut(B') \circ f \circ Aut(B)$ for any given f of type $FOO(B, B')$.

For example, our result that $Aut(B') \circ FOO(B, B') = FOO(B, B') \circ Aut(B) = FOO(B, B')$ is about sets. Given any $f_2 \in FOO(B, B')$ and $k \in Aut(B')$ there is *some* f_1 s.t. $f_2 = k \circ f_1$, but it needn't be the particular one given to us. The only exception is isomorphisms. Even with nonisomorphism coarsening morphisms, we aren't guaranteed that for a given f_1 and f_2 there exists some k and h for which $f_2 = k \circ f_1 \circ h$. This is more apparent if we note that $|Aut(B)| \neq |Aut(B')|$ in general, with equality guaranteed only if B and B' are isomorphic. The following examples illustrate this.

Ex. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $S' = \{a, b, c\}$, let $B = \{(1, 2), (3, 4, 5), (6), (7, 8), (9)\}$, and let $B' = \{(a), (b, c)\}$. Consider the automorphism $k : (a, b, c) \to (a, c, b)$. Suppose we have $f : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (b, b, c, c, c, a, c, c, a)$. It is easy to see that f is a surjective morphism from (S, B) to (S', B'). Clearly, $k \circ f = (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (c, c, b, b, b, a, b, b, a)$. We need $h \in Aut_B(S)$ s.t. $f \circ h = k \circ f$. However, f only can take 1 and 2 to b, so the only way $f \circ h$ can take $(3, 4, 5, 7, 8)$ to b is if h somehow maps all of them to $(1, 2)$, which is impossible for a bijection. This tells us that there need be no h corresponding to a given k.

Ex. let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $S' = \{a, b, c\}$, let $B = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$ (which we'll call (b_1, b_2, b_3)), let $B' = \{(a), (b, c)\}$, and let $f : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (a, a, a, b, b, b, c, c, c)$. Because the sizes of the two classes differ in B', there are only two automorphisms of B': Id_S and $(a, b, c) \to (a, c, b)$. There are $(3!) \cdot (3!)^3 = 1296$ automorphisms of B, but f filters these all into $3! = 6$ distinct $f \circ h$ functions. Of these, the only ones which map to a $k \circ f$ are the ones which don't mix b_1 with (b_2, b_3). I.e., 2 of the 6 permutations of the classes of B, and thus 432 of the 1296 overall automorphisms of B correspond to choices of $k \circ f$. The remaining 864 automorphisms of B filter through f into bijections of S' that are not representable via automorphisms of B' composed on f. As a concrete example, $h : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (4, 5, 6, 1, 2, 3, 7, 8, 9)$ yields $f \circ h : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (b, b, b, a, a, a, c, c, c)$, and neither possible automorphism k can effect that via $k \circ f$.

Nor does limiting ourselves to $PAut(B)$ and $PAut(B')$ help. It may seem like restricting ourselves to automorphisms that don't affect the class map should work, but that is not the case. In fact, our first example above applies to $PAut$ as well. In that example, $k : (a, b, c) \to (a, c, b)$ already is in $PAut(B')$, and we showed there isn't *any* automorphism in $Aut(B)$ that corresponds to it, let alone one in $PAut(B)$. It may be tempting to think that at least things work the other way, and that every $h \in PAut(B)$ yields some element of $PAut(B')$, even if in a non-injective non-surjective manner. This too fails, as the following example illustrates.

Ex. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $S' = \{a, b, c\}$, and let both B and B' be the trivial partitions. Consider $f : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (a, a, a, b, b, b, c, c, c)$ and $h : (1, 2, 3, 4, 5, 6, 7, 8, 9) \to (1, 4, 7, 2, 5, 8, 3, 6, 9)$. There is no k we can pick which corresponds to this because 1, 2, and 3 all map under h to different values, whereas under f they map to the same value. $f \circ h = (a, b, c, a, b, c, a, b, c)$. There is no $k \circ f$ that can replicate this.

Since our set-level proposition doesn't imply points (i)-(iii) above (and we've seen a couple of specific counterexamples), let's examine those three issues in their own rite. For each $FOO(B, B')$:

- (i) Given any $f_1, f_2 \in FOO(B, B')$ does there exist $k \in Aut(B')$ and $h \in Aut(B)$ s.t. $f_2 = k \circ f_1 \circ h$?
- (ii) Given any $f \in FOO(B, B')$, does $Aut(B') \circ f = f \circ Aut(B)$ (as sets)?
- (iii) Given any $f \in FOO(B, B')$, is $Aut(B') \circ f \circ Aut(B) = FOO(B, B')$?

Prop 2.53: Questions (i) and (iii) are equivalent.

Pf: (i)→ (iii): Suppose (i) holds. Fix f and pick any other $f_2 \in FOO(B, B')$. By (i), $f_2 = k \circ f \circ h$ for some k and h, so we have a way to get from f to f_2, and (iii) therefore holds.

Pf: (iii)→ (i): Suppose (iii) holds. Given f_2 and f_1, we know from (iii) that $FOO(B, B') = Aut(B') \circ f_1 \circ Aut(B)$. Since $f_2 \in FOO(B, B')$, there must be some k and h s.t. $f_2 = k \circ f_1 \circ h$. (i) therefore holds.

It turns out that the answers to these two equivalent questions, as well as to question (ii), are almost always 'no'. However, this doesn't mean that the concept they embody is of no interest. $Aut(B) \circ f \circ Aut(B')$ defines an equivalence relation, and thus partition, on $FOO(B, B')$.

We'll find the following result helpful:

Prop 2.54: Suppose f_1 and f_2 are coarsening morphisms from B to B'. By definition, they are isomorphisms from coarsenings B_1 and B_2 of B to B' and are isomorphisms from B to refinements B'_1 and B'_2 of B'.

- (i) B_1 and B_2 are isomorphic.
- (ii) B'_1 and B'_2 are isomorphic.
- (iii) $B_1 = B_2$ iff $f_2 = k \circ f_1$ for some $k \in Aut(B')$.
- (iv) $B'_1 = B'_2$ iff $f_2 = f_1 \circ h$ for some $h \in Aut(B)$.
- (v) f_1^{-1} is a coarsening morphism from B'_1 to both B_1 and B_2 (and thus an uncoarsening nonmorphism from B_1 and B_2 to B'_1). Likewise, f_2^{-1} is a coarsening morphism from B'_2 to both B_1 and B_2.
- (vi) If there exists a coarsening morphism from B to B' and if B'_1 is isomorphic to B', then there exists a coarsening morphism from B to B'_1.
- (vii) If there exists a coarsening morphism from B to B' and if B_1 is isomorphic to B, then there exists a coarsening morphism from B_1 to B'.

Pf: (i), (ii): f_1 and f_2 are isomorphisms from B_1 and B_2 to B', so $f_2^{-1} \circ f_1$ is an isomorphism from B_1 to B_2. Likewise, f_1 and f_2 are isomorphisms from B to B'_1 and B'_2 so $f_2 \circ f_1^{-1}$ is an isomorphism from B'_1 to B'_2.

Pf: (iii), (iv) (forward): If $B_1 = B_2$, then f_1 and f_2 are both isomorphisms between the same $B_1 = B_2$ and B'. From proposition 2.36, we therefore have that $f_2 = k \circ f_1$ for $k \in Aut(B')$ (or, equivalently, $f_1 \circ h$ for some $h \in Aut(B_1)$, but we don't care about that here). The same reasoning applies to f_1 and f_2 as isomorphisms from B to $B'_1 = B'_2$, and we get $f_2 = f_1 \circ h$ for $h \in Aut(B)$ (or, equivalently, $k \circ f_1$ for $k \in Aut(B'_1)$, but we don't care about that here).

Pf: (iii), (iv) (backward): Suppose $f_2 = k \circ f_1$ for some $k \in Aut(B')$. Since f_1 is an isomorphism from B_1 to B', $f_2 = k \circ f_1$ is an isomorphism from B_1 to B' as well. However, f_2 is an isomorphism from B_2 to B' and we know there is a unique isomorphic partner to B' under f_2, so $B_1 = B_2$. The exact same argument on the other end gives us (iv).

Pf: (v) f_1^{-1} is an isomorphism from B'_1 to B. Therefore, it is by definition a coarsening morphism from B'_1 to every coarsening of B, including B_1 and B_2.

Pf: (vi) Let f be a coarsening morphism from B to B' and let $k : B' \to B'_1$ be an isomorphism. Then f is an isomorphism from B to some refinement B'_R of B'. Since k is an isomorphism from B' to B'_1, proposition 2.12 part (ii) tells us that k is also an isomorphism from B'_R to some refinement B'_{1R} of B'_1. Therefore, $k \circ f$ is an isomorphism from B to B'_{1R} and therefore a coarsening morphism from B to B'_1.

Pf: (vii) Let f be a coarsening morphism from B to B' and let $h : B \to B_1$ be an isomorphism. Then f is an isomorphism from some coarsening B_C to B'. Since h is an isomorphism from B to B_1, proposition 2.12 part (i) tells us that h is also an isomorphism from B_C to some coarsening B_{1C} of B_1. Therefore, $f \circ h^{-1}$ is an isomorphism from B_{1C} to B' and therefore a coarsening morphism from B_1 to B'.

It may be tempting to hope that if $B'_1 = B'_2$ then $B_1 = B_2$, but this is not the case, as the following example shows.

Example with $B_1' = B_2'$ but $B_1 \neq B_2$: Let $S = S' = \{1,2,3,4,5,6\}$ and $B = \{(1,2),(3,4),(5,6)\}$ and $B' = \{(1,2,3,4),(5,6)\}$. $f_1 = Id_S$ is a coarsening morphism from B to B' with $B_1 = B'$ and $B_1' = B$. $f_2 : (1,2,3,4,5,6) \to (1,2,5,6,3,4)$ also is a coarsening morphism from B to B', with $B_2 = \{(1,2,5,6),(3,4)\}$ and $B_2' = \{(1,2),(5,6),(3,4)\} = B_1'$.

Although we saw that $NCMor(-,B) = NCMor(-,B') = NCMor(-,-) = Bij(S,S')$ (and ditto for $CMor$, $UMor$, and $NUMor$), it may seem that (vi) tells us that we can partition any given $NCMor(B,-)$ or $CMor(B,-)$ into isomorphism classes by B' (since it must contain whole isomorphism classes of this sort) and that (vii) does the same for $NCMor(-,B')$ and $CMor(-,B')$ and the isomorphism class of B. Unfortunately, this doesn't work. $NCMor(B,-)$ is the set of all coarsening morphisms from B to any partition, and it is true that if $NCMor(B,B') \neq \emptyset$ then $NCMor(B,B_1') \neq \emptyset$ for all B_1' isomorphic to B'. The problem isn't the isomorphism classes, it's the multiple hats a given f wears. A single $f \in NCMor(B,-)$ could be a member of many $NCMor(B,B')$ sets. For example, if $f \in NCMor(B,B')$ then $f \in NCMor(B,B_C')$ for every coarsening of B'. Successive proper coarsenings can't be isomorphic, so we have multiple isomorphism classes attached to the same f. We therefore can't partition $NCMor(B,-)$ by $[B']$, since a given f would be associated with multiple such classes.

What we *can* do is ask: for a given B, what partitions does there exist a coarsening morphism to? Denote this set $CP(B)$, and denote by $RP(B')$ the set of partitions from which a coarsening morphism exists to B'. Formally, $CP(B) = \{B' \in Par(); CMor(B,B') \neq \emptyset\}$ and $RP(B') \equiv \{B \in Par(); CMor(B,B') \neq \emptyset\}$. Then (vi) and (vii) tell us that $CP(B)$ and $RP(B')$ respect equivalence classes, so we can define $CP'(B)$ and $RP'(B')$ quotiented by isomorphism class [[]] (since we're working in $Par()$). In fact, (vi) and (vii) also tell us that we have a notion of $CP([B])$ and $RP([B'])$. Note that we can do something similar if we confine ourselves to a given S and S', and quotient by the relevant [] on each end. CP and RP define maps from $Par'()$ (or $Par'(S)$). However, we must be careful. We do indeed have two natural maps from $Par'()$ (or $Par'(S)$), but to what? They are not maps *to* $Par'()$ (or $Par'(S')$). Each $CP(B)$ and $RP(B')$ is a set of partitions, and each $CP'(B)$, $RP'(B')$, $CP'([B])$, and $RP'([B'])$ is a *set* of isomorphism classes. I.e., we have a multi-valued function (aka a function to $2^{Par'()}$ (or $2^{Par'(S')}$)). Nor can we speak of CP and RP (or CP' and RP') as being inverses. Yes, $[B'] \in CP'([B])$ iff $[B] \in RP'([B'])$, but they need not be the only elements of those sets. Only for isomorphisms (the simplest coarsening morphisms), do we get a natural bijection. We'll have more to say about that shortly.

At this point, it is useful to mention that if B_R is a refinement of B on the same S, there is no guaranteed relationship between $Aut(B)$ and $Aut(B_R)$, although $PAut(B_R) \subseteq PAut(B)$ (with equality iff $B = B_R$).

($Aut(B_R) \subset Aut(B)$): Let $S = \{1,2,3,4\}$ and $B_R = \{(1,2),(3,4)\}$ and $B = \{(1,2,3,4)\}$. Then $Aut(B) = PAut(B)$ consists of all 24 permutation maps. However, $Aut(B_R)$ consists of only 8 maps and $PAut(B_R)$ consists of only 4.

($Aut(B_R)$ not a subset of $Aut(B)$): Let $S = \{1,2,3,4,5,6\}$ and $B_R = \{(1,2),(3,4),(5,6)\}$ and $B = \{(1,2,3,4),(5,6)\}$. Then $f : (1,2,3,4,5,6) \to (1,2,5,6,3,4)$ is in $Aut(B_R)$ but not $Aut(B)$. In fact, $Aut(B_R)$ and $Aut(B)$ both happen to have 48 elements in this example. However, they are not the same 48 elements. For example, $f_1 : (1,2,3,4,5,6) \to (1,3,2,4,5,6)$ is in $Aut(B)$ but not $Aut(B_R)$.

Prop 2.55: Given (S,B) and (S',B'), define $f_1 \sim_1 f_2$ (for $f_1, f_2 \in Map(S,S')$) iff $f_2 = k \circ f_1 \circ h$ for some $k \in Aut(B')$ and $h \in Aut(B)$, and define $f_1 \sim_2 f_2$ iff this holds with $k \in PAut(B')$ and $h \in PAut(B)$. Then:

We don't really care about the whole of $Map(S,S')$, but defining $\sim_1$ and $\sim_2$ on it proves convenient because every one of our $FOO(B,B')$ sets is a subset of $Map(S,S')$.

- (i) $\sim_1$ is an equivalence relation on $Map(S,S')$.
- (ii) $\sim_2$ is an equivalence relation on $Map(S,S')$.
- (iii) $\sim_2$ is a refinement of $\sim_1$.
- (iv) Every $FOO(B,B')$ respects the classes of $\sim_1$ (and thus of $\sim_2$) in the sense that it is a union of whole classes.
- (v) $\sim_1$ is guaranteed to induce the trivial partition (i.e. every $f_1 \sim_1 f_2$) only for $Iso(B,B')$.

  It follows that if B and B' are isomorphic, then $UMor(B,B') = CMor(B,B') = Iso(B,B')$, $NUMor(B,B') = NCMor(B,B') = \emptyset$, and everyone else has at least two classes (unless they coincidentally equal $Iso(B,B')$). For B and B' not isomorphic, $Iso(B,B') = \emptyset$, and the other FOO's may have zero, one, or multiple classes.

- (vi) $Aut(B) \circ f = f \circ Aut(B')$ is guaranteed only for $Iso(B,B')$.

Although we define $\sim_1$ and $\sim_2$ on $Map(S, S')$, we don't care about most of the equivalence classes. For the B and B' in question, the largest sets we care about are $Mor(B, B')$ (for morphisms) and $NUMor(B', B)$ (for nonmorphisms). However, item (iv) tells us that it doesn't matter. $Map(S, S')$ is allowed as a convenience, because each of our sets $FOO(B, B')$ contains only whole classes. Put another way, if f is of type FOO (i.e. $f \in FOO(B, B')$), then the entire class of f under $\sim_1$ is in $FOO(B, B')$.

Pf: (i) We have reflexivity from $f = Id_{S'} \circ f \circ Id_S$. If $f_2 = k \circ f_1 \circ h$, then $f_1 = k^{-1} \circ f_2 \circ h^{-1}$, so we have symmetry. If $f_2 = k_1 \circ f_1 \circ h_1$ and $f_3 = k_2 \circ f_2 \circ h_2$, then $f_3 = k_2 \circ k_1 \circ f_1 \circ h_1 \circ h_2$, so we have transitivity.

Pf: (ii) The same proof holds for $PAut$, since $Id_S \in PAut(B)$ and $PAut(B)$ is a group just like $Aut(B)$.

Pf: (iii) Suppose $f_2 = k \circ f_1 \circ h$ for $k \in PAut(B')$ and $h \in PAut(B)$. Since $PAut \subseteq Aut$, $k \in Aut(B')$ and $h \in Aut(B)$, and the same expression satisfies the $\sim_1$ condition. I.e. $f_2 \sim_2 f_1 \implies f_2 \sim_1 f_1$, and $\sim_2$ refines $\sim_1$.

Pf: (iv) This follows immediately from proposition 2.52, part (ii). Suppose $f_2 = k \circ f_1 \circ h$, with $f_1 \in FOO(B, B')$ and $k \in Aut(B')$ and $h \in Aut(B)$. Then $f_1 \circ h \in FOO(B, B') \circ Aut(B)$, which we know from proposition 2.52 is in $FOO(B, B')$. We then also know (again from proposition 2.52) that $k \circ (f_1 \circ h) \in FOO(B, B')$. Therefore $f_2 \in FOO(B, B')$. This means that if $f_1 \in FOO(B, B')$, then $Aut(B') \circ f_1 \circ Aut(B) \subseteq FOO(B, B')$, and thus $FOO(B, B')$ is comprised of a union of such classes.

Pf: (v) for Iso: Either $Iso(B, B')$ is empty (in which case it vacuously satisfies the condition) or proposition 2.36 tells us that any two isomorphisms are related by a member of $Aut(B)$ or, equivalently, a member of $Aut(B')$. Therefore, Iso has a single class under $\sim_1$.

(v) counterexample for $MapTo$ (and thus $SMor$, Mor, and $FMapTo$): Let $S = \{1, 2, 3, 4, 5, 6\}$, let $S' = \{1, 2\}$, let $B = \{(1, 2, 3, 4, 5, 6)\}$ and let $B' = \{(1, 2)\}$ (i.e. both are trivial partitions). Suppose $f_1 : (1, 2, 3, 4, 5, 6) \rightarrow (1, 1, 1, 2, 2, 2)$ and $f_2 : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 2, 2, 2, 2)$. $Aut(B)$ and $Aut(B')$ consist of permutations. Any permutation of B merely changes which 3 elements f_1 maps to 1, and any permutation of B' merely swaps which element of B' a given set of 3 elements of B is mapped to by f_1. There is no $k \circ f_1 \circ h$ which can change from a 3-3 split to a 1-5 split. However, both are maps 'to'. Therefore, our $MapTo(B, B')$ contains more than one $\sim_1$ class.

(v) counterexample for $NCMor$ (and thus $CMor$, $NUMor$, and $UMor$): We can adapt our earlier $NCMor$ counterexample. Let $S = S' = \{1 \dots 16\}$, let B contain 4 classes of sizes $|b_1| = 2$, $|b_2| = 3$, $|b_3| = 4$, and $|b_4| = 7$, and let B' contain 2 classes of sizes $|b_1'| = 7$ and $|b_2'| = 9$. Since no two classes of B are the same size, $Aut(B) = PAut(B)$, and the same is true of B'. The only automorphisms on either end involve internal permutations of each class. Let f_1 take b_4 to b_1' and merge b_1, b_2, and b_3 into b_2'. Let f_2 merge b_2 and b_3 into b_1' and merge b_1 and b_4 into b_2'. We cannot write $f_2 = k \circ f \circ h$ for any $k \in PAut(B') = Aut(B')$ and $h \in PAut(B) = Aut(B)$ because none of them will change from a $1-3$ split to a $2-2$ split. Our $NCMor(B, B')$ therefore has at least two classes under $\sim_1$. We saw that $CMor = Iso$ iff B and B' are isomorphic, and otherwise $CMor = NCMor$. In the present case, $CMor = NCMor$ and has at least two classes. Now, consider $UMor(B', B) = NUMor(B', B) = NCMor(B, B')^{-1}$. $f_2^{-1} = h^{-1} \circ f^{-1} \circ k^{-1}$ iff $f_2 = k \circ f \circ h$, so the inverses respect the classes (with the appropriate $Aut(B) \leftrightarrow Aut(B')$ swap). This means that $NUMor(B', B) = UMor(B', B)$ for our example has at least two classes as well.

Comment on (v) for $NCMor$, et al: Since each element of $NCMor(B, B')$ is an isomorphism from B to some B_R' and an isomorphism from some B_C to B', we may imagine that that $\sim_1$ divvies up $NCMor(B, B')$ into classes indexed by the choice of B_R', or perhaps by the choice of B_C. We already know from proposition 2.54 that the relevant B_R''s all are isomorphic and the relevant B_C's all are isomorphic, so this seems plausible. Unfortunately, $\sim_1$ doesn't act so simply on $NCMor$. Each $Aut(B') \circ f$ wholly resides within a single class, as does each $f \circ Aut(B)$. However, neither typically fills the class in which it lives. A given $Aut(B') \circ f \circ Aut(B)$ combines different B_C's and different B_R''s. In the comments and example following proposition 2.54 we saw that coarsening morphisms f_1 and f_2 from B to B' can have the same B_R' but differing B_C's, and it is also obvious that we can have the same B_C but different B_R''s. Therefore, a given class can mix multiple B_C's and multiple B_R''s. The following example illustrates this.

Example of a single $\sim_1$ class with multiple B_R''s and B_C's: Returning to the example below proposition 2.54, let $S = S' = \{1, 2, 3, 4, 5, 6\}$ and $B = \{(1, 2), (3, 4), (5, 6)\}$ and $B' = \{(1, 2, 3, 4), (5, 6)\}$. $f_1 = Id_S$ is a coarsening morphism from B to B' (with $B_1 = B'$ and $B_1' = B$), so let's consider $Aut(B') \circ Id_S \circ Aut(B) = Aut(B') \circ Aut(B)$ (this expression only makes sense since $S = S'$, of course). We saw that $f_2 : (1, 2, 3, 4, 5, 6) \rightarrow (1, 2, 5, 6, 3, 4)$ is a coarsening morphism from B to B', with $B_2 = \{(1, 2, 5, 6), (3, 4)\}$ and $B_2' = \{(1, 2), (5, 6), (3, 4)\} = B_1'$. Moreover, $f_2 \in Aut(B)$ and thus can be expressed as $Id_{S'} \circ f_1 \circ f_2 = f_2$. On the other hand, consider $f_3 = (1, 2, 3, 4, 5, 6) \rightarrow (1, 3, 2, 4, 5, 6)$. This is a coarsening morphism from B to B' as well. In this case, $B_3 = B'$ and $B_3' = \{(1, 3), (2, 4), (5, 6)\}$. Since $f_3 \in Aut(B')$, it may be written $f_3 = f_3 \circ f_1 \circ Id_S = f_3$. We therefore have that f_1, f_2, and f_3 are all in the same $\sim_1$ class, but between them we have distinct B_C's (B_1 vs B_2) and distinct B_R''s (B_1' vs B_3').

Pf: (vi) For Iso, proposition 2.36 gives us this.

(vi) counterexample for $MapTo$ (and thus $SMor$, Mor, and $FMapTo$): Let $S = \{1, 2, 3, 4\}$ and $S' = \{1, 2\}$, and let both B and B' be the trivial partitions. There are 14 surjective maps from S to S', and all of them are maps from B 'to' B'. Consider $f : (1, 2, 3, 4) \rightarrow (1, 1, 2, 2)$. $Aut(B) = PAut(B)$ has 24 elements and $Aut(B') = PAut(B')$ has 2 elements. $Aut(B') \circ f$ consists of f and $f_1 : (1, 2, 3, 4) \rightarrow (2, 2, 1, 1)$. On the other hand, $f \circ Aut(B)$ consists of f and f_1 (via $f \circ ((1, 2, 3, 4) \rightarrow (3, 4, 1, 2))$) and $f_2 : (1, 2, 3, 4) \rightarrow (1, 2, 1, 2)$ (via $f \circ ((1, 2, 3, 4) \rightarrow (1, 3, 2, 4))$) and numerous others. Note that $Aut(B') \circ MapTo(B, B') = MapTo(B, B') \circ Aut(B) = MapTo(B, B')$. However, for any given f, $Aut(B') \circ f$ need not equal $f \circ Aut(B)$.

(vi) counterexample for $NCMor$ (and thus $CMor$, $NUMor$, and $UMor$): Let $S = S' = \{1,2,3,4\}$ and $B = \{(1,2,3),(4)\}$ and $B' = \{(1,2,3,4)\}$. Since the class sizes differ, $Aut(B) = PAut(B)$ has $3! = 6$ elements. $Aut(B') = PAut(B')$ has $4! = 24$ elements. Every bijection from S to S' is a coarsening morphism, and there are $4! = 24$ of these. Consider $f = Id_S$. $f \circ Aut(B) = NCMor(B,B')$ has all 24 elements. However, $Aut(B') \circ f$ has at most 6 elements. Therefore, the two cannot be equal.

A couple of notes on the (vi) counterexample for $NCMor$: (a) For any isomorphism, $Aut(B') \circ f = f \circ Aut(B)$. However, this doesn't apply here. f is an isomorphism from B to some B'_R or from some B_C to B'. $Aut(B) \approx Aut(B'_R)$ and $Aut(B') \approx Aut(B_C)$, but we do *not* have $Aut(B) \approx Aut(B')$. (b) We may be tempted to assume that a coarsening morphism induces a homomorphism from $Aut(B)$ to a subgroup of $Aut(B')$ and that $f \circ Aut(B) \subseteq Aut(B') \circ f$ in general. However, this is not the case either. We saw earlier that, even for refinements on a single S, $Aut(B_R)$ need have no specific relation to $Aut(B)$. As a result, there is no reason to expect a relationship between f composed on them.

Define $FOO'(B,B') \equiv FOO(B,B')/{\sim_1}$ to be the set of $\sim_1$ classes in $FOO(B,B')$. Proposition 2.55, part (iv) guarantees that this is well-defined. The following then holds:

Bear in mind that these quotients are just sets, not groups. We could take a quotient by $\sim_2$ as well, but doing so offers no extra insight for our present purposes.

Prop 2.56: The following relationships hold for our $FOO'(B,B')$ sets:

- (i) $Iso'(B,B') \subseteq [CMor'(B,B'), MapTo'(B,B')] \subset SMor'(B,B') \subset Mor'(B,B')$.
- (ii) $Iso'(B,B') \subseteq MapTo'(B,B') \subset FMapTo'(B,B') \subset Mor'(B,B')$.
- (ii) $NCMor'(B,B') \subset CMor'(B,B') \subset SMor'(B,B') \subset Mor'(B,B')$.
- (iii) $NUMor'(B,B') \subset UMor'(B,B')$.
- (iv) $Iso'(B,B') = UMor'(B,B') \cap CMor'(B,B')$.
- (v) $NCMor'(B,B') \cap NUMor'(B,B') = \emptyset$.
- (vi) If $|S| > |S'|$ then $CMor'(B,B') = UMor'(B,B') = NCMor'(B,B') = NUMor'(B,B') = Iso'(B,B') = \emptyset$.
- (vii) If $|S| < |S'|$, then all but $Mor'(B,B')$ and $FMapTo'(B,B')$ are $\emptyset$.
- (viii) If $|B| < |B'|$ then all but $UMor'(B,B')$ and $NUMor'(B,B')$ are $\emptyset$.
- (ix) If $|B| > |B'|$ then $Iso'(B,B') = MapTo'(B,B') = FMapTo'(B,B') = UMor'(B,B') = NUMor'(B,B') = \emptyset$.
- (x) If B and B' are isomorphic, then:
  - ◇ (a) $|NCMor'(B,B')| = |NUMor'(B,B')| = 0$.
  - ◇ (b) $|CMor'(B,B')| = |UMor'(B,B')| = |Iso'(B,B')| = 1$.
  - ◇ (c) $|FMapTo'(B,B')| \geq |MapTo'(B,B')| \geq 1$.
  - ◇ (d) $|Mor'(B,B')| \geq |SMor'(B,B')| \geq 1$.
- (xi) If $|S| \geq |S'|$ and $|B| \geq |B'|$ and B is not isomorphic to B', then:
  - ◇ (a) $|Iso'(B,B')| = 0$.
  - ◇ (b) $|CMor'(B,B')| = |NCMor'(B,B')| > 0$ or $|UMor'(B,B')| = |NUMor'(B,B')| > 0$ or both are zero. I.e., both can't be > 0.
  - ◇ (c) $|Mor'(B,B')| > 0$ and $|SMor'(B,B')| \geq 0$.
  - ◇ (d) If $|B| = |B'|$, then $|FMapTo'(B,B')| > 0$ and $|MapTo'(B,B')| \geq 0$. Otherwise, both are 0.

Pf: (i)-(v) follow immediately from the corresponding relations amongst the $FOO(B,B')$ sets and the fact that all the FOO sets respect the $\sim_1$ classes.

Pf: (vi)-(ix) follow immediately from the corresponding relations in proposition 2.50.

Pf: (x) (a) and (b) follow directly from the fact that $Iso'(B, B')$ has a single class. The rest follow directly from the inclusions in (i)-(iii).

Note on the inequalities in (x) part (c): When B and B' are isomorphic, $MapTo'(B, B') = Iso'(B, B')$ iff the only bijective g's induced by surjective f's are induced by bijective f's. This happens iff all the classes are finite (although the number of classes can be infinite). [For a set X s.t. $|X| \geq \aleph_0$, we always can find a surjective but noninjective map from X to itself.] Otherwise, $|MapTo'(B, B')| > 1$ because it contains and is strictly larger than $Iso'(B, B')$. On the other hand $|FMapTo'(B, B')| > 1$ almost always. The only exception is when B' is the singleton partition, in which case $FMapTo(B, B') = MapTo(B, B') = Iso(B, B')$ and ditto for the primed versions. We can construct an element of $FMapTo'(B, B')$ which is not in $MapTo'(B, B')$ from any non-singleton b' class by simply changing a surjective class-to-class map to be nonsurjective. Therefore, $MapTo(B, B')$ is a proper subset of $FMapTo(B, B')$ when B' isn't the singleton partition. Since the $\sim_1$-classes are preserved, this means that $MapTo'(B, B')$ is a proper subset of $FMapTo'(B, B')$ as well. Note that B' is the singleton partition iff B is the singleton partition, since we assumed they're isomorphic.

Note on the inequalities in (x) part (d): If B' (and thus B) is a finite singleton partition, the only surjective maps are bijective, and no non-surjective map induces a surjective g, so we only have isomorphisms. In that case, $Mor = SMor = Iso$. For an infinite singleton partition, we can map multiple classes to the same class, but there is no room to introduce non-surjectivity of f, so $|Mor'| = |SMor'| > 1$. If B' is finite with only finite-sized classes, then f must be bijective or it can't be surjective. However, we can introduce nonsurjectivity of f into any of the non-singleton classes without harming the surjectivity of the induced g. Therefore, $|Mor'| > |SMor'| = |Iso'| = 1$. If the number of classes is infinite (and some are non-singleton), then we can have a non-injective but surjective f that induces a surjective g. We also can introduce non-surjectivity into any non-singleton class, so $|Mor'| > |SMor| > 1$.

Pf: (xi) (a) holds by definition. (b) follows from proposition 2.49. The first inequality in (c) follows from the fact that we can always find a morphism when $|B| \geq |B'|$. (d) partly follows from the fact that we can always find a morphism such that the induced g is bijective when $|B| = |B'|$, and the rest follows from proposition 2.50.

Ex. where $|B| \geq |B'|$ and $|S| \geq |S'|$ but $MapTo(B, B') = SMor(B, B') = \emptyset$: Let $S = S' = \{1, 2, 3, 4\}$ and let $B = \{(1, 2), (3, 4)\}$ and $B' = \{(1), (2, 3, 4)\}$. There is no surjective $f : S \to S'$ that is a morphism, let alone a 'map to'. However, $f : (1, 2, 3, 4) \to (1, 1, 2, 3)$ is a flexible map 'to' B' from B, and thus a morphism as well.

2.13. **How morphisms interact with our classes of partitions.**

Consider $FOO(B, B')$ and $FOO'(B, B')$. These consist of maps and classes of maps from S to S'. A natural question to ask is how they behave under isomorphic changes in B and B'. Suppose $B_1 \sim B$ and $B'_1 \sim B'$. Is there a relationship between $FOO(B, B')$ and $FOO(B_1, B'_1)$ and/or between $FOO'(B, B')$ and $FOO'(B_1, B'_1)$?

Note that we must ask this about individually about the change from B to B_1 and from B' to B'_1, because for all but $Iso(B, B')$, the two arguments play asymmetric roles.

We know that each bijection $f : S \to S'$ induces a bijection between $Par(S)$ and $Par(S')$, partnering isomorphic pairs under f. We saw that this respects isomorphism-classes, and that if B and B' are isomorphic partners under f, then f partners every element of $[B]$ with an element of $[B']$ and vice versa. I.e., it induces a bijection between $Par'(S)$ and $Par'(S')$ as well as bijections between each $[B]$ and its partner $[B']$. Most important, we learned that the induced bijection between $Par'(S)$ and $Par'(S')$ is independent of f and is "natural" in this sense.

Formally, we saw that all $Push_f$'s for bijective f's induce the same map between $Par'(S)$ and $Par'(S')$ — thus making it natural. Likewise, all $Pull_f$'s for bijective f's induce the same map between $Par'(S')$ and $Par'(S)$. These two natural maps are bijections and inverses of one another.

Each individual class and its partner are bijective as well, but there is no natural choice of bijection. Any given bijective $f : S \to S'$ supplies us with one.

This means that we have cause to hope that the bijection-based FOO's — Iso, $CMor$, $NCMor$, $UMor$, and $NUMor$ — behave well vis-a-vis isomorphism classes of partitions.

We've also seen that for a *surjective* f, $Push_f$ is surjective but not necessarily injective and $Pull_f$ is injective but not necessarily surjective. By proposition 2.25, $(Push_f \circ Pull_f)(B') = B'$ but $(Pull_f \circ Push_f)(B) \geq B$ (i.e. we get a coarsening of B).

Our maximally refined and maximally coarsened partners don't constitute a bijection between $Par(S)$ and $Par(S')$. For $|S| \geq |S'|$, this is no surprise, since a bijection between $Par(S)$ and $Par(S')$ doesn't exist. If $|S| = |S'|$, $Par(S)$ and $Par(S')$ are bijective, but a surjective f doesn't induce a bijection between them.

The following proposition tells us some relationships between $FOO(B, B')$ and $FOO(B_1, B'_1)$ and between $FOO'(B, B')$ and $FOO'(B_1, B'_1)$ for $B \sim B_1$ and $B' \sim B'_1$:

Prop 2.57: Let $B \sim B_1$ and $B' \sim B'_1$, and let FOO be any of our nine types. Then:

- (i) $f \in FOO(B, B')$ iff for any $h \in Iso(B, B_1)$ and $k \in Iso(B', B'_1)$, $l \equiv k \circ f \circ h^{-1}$ is in $FOO(B_1, B'_1)$.
- (ii) $FOO(B, B') = Iso(B'_1, B') \circ FOO(B_1, B'_1) \circ Iso(B, B_1)$.
- (iii) Any fixed choice of h and k creates a bijection $\alpha_{h,k}$ between $FOO(B, B')$ and $FOO(B_1, B'_1)$, given by $f \to k \circ f \circ h^{-1}$.
- (iv) $\alpha_{h,k}$ respects $\sim_1$ on both ends. I.e., $f \sim_1 f_1$ iff $\alpha_{h,k}(f) \sim_1 \alpha_{h,k}(f_1)$.

  Bear in mind that $\sim_1$ is distinct on the left and the right. $f \sim f_1$ speaks of whether $f \in Aut(B') \circ f \circ Aut(B)$, with $f, f_1 : S \to S'$. On the other hand, $\alpha_{h,k}(f) \sim_1 \alpha_{h,k}(f_1)$ speaks of whether $\alpha_{h,k}(f) \in Aut(B'_1) \circ \alpha_{h,k}(f_1) \circ Aut(B_1)$, with $\alpha_{h,k}(f), \alpha_{h,k}(f_1) : S_1 \to S'_1$. This is one of the rare times that the $\sim_1$ notation's implicit dependences can lead to confusion.
- (v) The induced bijection $\alpha'_{h,k} : FOO'(B, B') \to FOO'(B_1, B'_1)$ is independent of h and k.

I.e., we have a natural bijection between $FOO'(B, B')$ and $FOO'(B_1, B'_1)$ and a non-natural bijection between $FOO(B, B')$ and $FOO(B_1, B'_1)$ which is locked down by a choice of h and k. Note, however, that in (iii) we are not claiming that distinct choices of h and k produce distinct $\alpha_{h,k}$'s. For example, consider Iso, in which case all the Aut's and Iso's are the same size. Suppose this size is some finite n. Then there are n choices of l, but n^3 choices of (h, f, k). Obviously, many of the $k \circ f \circ h$ combos must result in the same l.

Pf: (i) General considerations: For any FOO, consider some $f \in FOO(B, B')$. Pick some k and h, and let $l \equiv k \circ f \circ h^{-1}$. If we prove that $l \in FOO(B_1, B'_1)$ (the "if"), then the converse (the "iff") is automatic, because we just swap the roles of B and B_1 and swap the roles of B' and B'_1. Then, we start with l, and $f = k^{-1} \circ l \circ h$, with $k^{-1} \in Iso(B'_1, B')$ and $h^{-1} \in Iso(B_1, B)$. From the already-proved forward direction, it then follows that if $l \in FOO(B_1, B'_1)$, then $f \in FOO(B, B')$. We therefore need only prove the forward direction.

Pf: (i) Iso: Suppose $f \in Iso(B, B')$. Then pick any k and h and define $l \equiv k \circ f \circ h^{-1}$. Since these are all isomorphisms, $l \in Iso(B_1, B'_1)$.

Pf: (i) $CMor$, $NCMor$: Suppose $f \in NCMor(B, B')$. Pick some k and h. As a coarsening morphism, $f \in Iso(B, B'_R)$ for some refinement B'_R of B'. Since $h \in Iso(B, B_1)$, $f \circ h^{-1} \in Iso(B_1, B'_R)$. Since k is an isomorphism from B' to B'_1, we know from proposition 2.12 that it also is an isomorphism from B'_R to some refinement B'_{1R} of B'_1. Therefore, $k \circ f \circ h^{-1} \in Iso(B_1, B'_{1R})$. This characterizes it as a coarsening morphism from B_1 to B'_1. Since $CMor$ equals either $NCMor$ or Iso, our proofs for those hold for it too.

Pf: (i) $UMor$, $NUMor$: Suppose $f \in NUMor(B, B')$. Then $f^{-1} \in NCMor(B', B)$. By the proof for $NCMor$, this means that $a \circ f^{-1} \circ c^{-1} \in NCMor(B'_1, B_1)$ for any $c \in Iso(B', B'_1)$ and $a \in Iso(B, B_1)$. This means that $(a \circ f^{-1} \circ c^{-1})^{-1} = c \circ f \circ a^{-1}$ is in $NUMor(B_1, B'_1)$. It is of the form we need, with $h = a$ and $k = c$. $UMor$ follows for the same reason as $CMor$ did.

Pf: (i) Mor: Let $f \in Mor(B, B')$. Then every $f(b) \subseteq b'$ for some b', and the resulting g is surjective. Pick any k and h, and define $l \equiv k \circ f \circ h^{-1}$. Consider $l(b_1)$ for some $b_1 \in B_1$. We need to prove that it is $\subseteq$ to some b'_1 and that the resulting g_1 (induced by l) is surjective. Since h is an isomorphism, $h^{-1}(b_1) = b$ for some $b \in B$. Since f is a morphism, $f(b) \subseteq b'$ for some $b' \in B'$. Since k is an isomorphism, $k(b') = b'_1$ for some $b'_1 \in B'_1$. Therefore, $l(b_1) = k(f(h^{-1}(b_1))) = k(f(b)) \subseteq k(b') = b'_1$, and we have $l(b_1) \subseteq b'_1$ as desired. Now, let's show that the corresponding induced g is surjective. Consider some $b'_1 \in B'_1$. $k^{-1}(b'_1) = b'$ for some $b' \in B'$. Since f is a morphism from B to B', $f^{-1}(b') = \cup b_i$ for some nonempty union of b_i's in B. $h(\cup b_i) = \cup h(b_i) = \cup b_{1i}$, a nonempty union of classes of B_1. So $l^{-1}(b'_1)$ is nonempty, and the induced g is surjective. Therefore, $l \in Mor(B_1, B'_1)$.

Pf: (i) $SMor$: From the proof for Mor, we know that l is a morphism since f is. Since f is surjective, and k and h are isomorphisms (and thus bijections), all three of k, f, and h^{-1} are surjective, so their composition l is surjective too. This makes it a surjective morphism, and $l \in SMor(B_1, B'_1)$.

Pf: (i) $FMapTo$: From the proof for Mor, we know that l is a morphism since f is. The g induced by f is a bijection. We already know that the g_1 induced by l is surjective, so let's show that it is injective. Since k is an isomorphism, $k^{-1}(b'_1) = b'$ for some b'. $f^{-1}(b') = \cup b_i$ for some nonempty union of b_i's in B. However, since the induced g is bijective, this union has one and only one entry. Therefore, $f^{-1}(b') = b$ for some b. Since h is an isomorphism, $h(b) = b_1$ for some $b_1 \in B_1$. Therefore, $l^{-1}(b'_1) = b_1$ is a unique class, and the induced g_1 is injective (and thus bijective). l therefore is a flexible map 'to' and is in $FMapTo(B_1, B'_1)$.

Pf: (i) $MapTo$: We adapt the proof of $SMor$ to $MapTo$ in exactly the same way we adapted the proof of Mor to $FMapTo$.

Pf: (ii) This follows from the 'iff' in (i).

Pf: (iii) We already exhibited this bijection in our proof of (i). That it is a bijection is trivial from the definition, since it is invertible.

Pf: (iv) We now need to be careful with our alphabet since we used h and k for both our $\sim_1$ and our $\alpha_{h,k}$ definitions. We'll use h and k in the latter sense and use a and b for the former. Pick some h and k, and let $f_1 \sim_1 f_2$. By the definition of $\sim_1$, $f_2 = a \circ f_1 \circ b$ for some $a \in Aut(B')$ and $b \in Aut(B)$. By the definition of $\alpha_{h,k}$, $\alpha_{h,k}(f_1) = k \circ f_1 \circ h^{-1}$ and $\alpha_{h,k}(f_2) = k \circ f_2 \circ h^{-1}$. We wish to show that $(k \circ f_2 \circ h^{-1}) = a_1 \circ k \circ f_1 \circ h^{-1} \circ b_1$ is true for some $a_1 \in Aut(B'_1)$ and $b_1 \in Aut(B_1)$. The left side is $k \circ a \circ f_1 \circ b \circ h^{-1}$. Pick $a_1 \equiv k \circ a \circ k^{-1}$ and $b_1 \equiv h \circ b \circ h^{-1}$. Since k, h, a, and b are isomorphisms, their inverses and compositions are too. $k \circ a \circ k^{-1}$ takes B'_1 to B' to B' to B'_1, and thus $a_1 \in Aut(B'_1)$. Similarly, $h \circ b \circ h^{-1}$ takes B_1 to B to B to B_1, and thus $b_1 \in Aut(B_1)$. Plugging this a_1 and b_1 into the right side of our desired equation, we get $a_1 \circ k \circ f_1 \circ h^{-1} \circ b_1 = k \circ a \circ k^{-1} \circ k \circ f_1 \circ h^{-1} \circ h \circ b \circ h^{-1} = k \circ a \circ f_1 \circ b \circ h^{-1}$. This is equal to the expression we obtained for the left side, so our equation holds. Therefore, $\alpha_{h,k}(f_1) \sim_1 \alpha_{h,k}(f_2)$. Since $\alpha_{h,k}$ is invertible, the same argument works the other way, and we have an iff.

Pf: (v) We established in (iv) that any given $\alpha_{h,k}$ respects $\sim_1$ on both ends. Denoting by $[f]$ the $\sim_1$ class containing f, (iv) tells us that $f_1, f_2 \in [f]$ iff $\alpha_{h,k}(f_1), \alpha_{h,k}(f_2) \in [\alpha_{h,k}(f)]$, where we note (as in our earlier comment) that $\sim_1$ and $[]$ for f, f_1, f_2 are implicitly relative to B, B' and $\sim_1$ and $[]$ for $\alpha_{h,k}(*)$ are implicitly relative to B_1, B'_1. This tells us that $\alpha_{h,k}$ creates a bijection between classes of $FOO(B, B')$ and classes of $FOO(B_1, B'_1)$. I.e., a bijection between $FOO'(B, B')$ and $FOO'(B_1, B'_1)$. To show that this is independent of the choice of h and k, it suffices to show that $\alpha_{h,k}(f) \sim_1 \alpha_{h_1,k_1}(f)$ for any $f \in FOO(B, B')$ and any choices of h, h_1, k, and k_1. In that case, changing h and k doesn't alter the $\sim_1$ class we end up in. To prove what we need, we must show that $k_1 \circ f \circ h_1^{-1} = a \circ k \circ f \circ h^{-1} \circ b$ for some $a \in Aut(B'_1)$ and $b \in Aut(B_1)$. Just pick $a = k_1 \circ k^{-1}$ and $b = h \circ h_1^{-1}$, and our desired equation holds. Since isomorphisms compose and invert, both a and b are isomorphisms. a takes B'_1 to B' to B'_1, and thus is in $Aut(B'_1)$. b takes B_1 to B to B_1, and thus is in $Aut(B_1)$. We therefore have accomplished our goal.

2.14. **A natural partition of $Bij(S, S')$ or $Surj(S, S')$?** For a given B, we know that every $f \in Bij(S, S')$ takes B to a unique isomorphic partner, and for a given B', we know that every $f \in Bij(S, S')$ takes a unique isomorphic partner to B'. Any two f's taking the same B to the same B' are related by an element of $Aut(B)$ or, equivalently, of $Aut(B')$.

If we pick B, then we can partition $Bij(S, S')$ by the relevant isomorphic partner B'. Taking liberties with our notation, we can write this as $Bij(S, S')/Aut(B)$. Similarly, if we pick B', then we can partition $Bij(S, S')$ by the relevant isomorphic partner B. We can write this as $Bij(S, S')/Aut(B')$.

It may be tempting to think that this allows us to partition $Bij(S, S')$ into classes by $Iso(B, B')$ (or perhaps sets of identical $Iso(B, B')$'s). Unfortunately, this is not the case. If we pick a given B and B', then we have two partitions as described. Their common refinement is a partition too, and $Iso(B, B')$ is a class of that partition. However, it's the only Iso that serves in this capacity. Any other class of the refinement is of the form $Iso(B, B'_1) \cap Iso(B_1 \cap B')$ for some partitions B_1 of S and B'_1 of S'. Such an intersection needn't equal $Iso(A, C)$ for any partitions A of S and C of S'.

If we think of $Bij(S, S')/Aut(B)$ has partitioning $Bij(S, S')$ into horizontal stripes, and $Bij(S, S')/Aut(B')$ as partitioning it into vertical ones, the resulting grid contains $Iso(B, B')$ as one square, and no other square need be of the form $Iso(A, C)$. Moreover, these three partitions of $Bij(S, S')$ depend on our choice of B and B'. A different B and B' yield a different grid, with a different special $Iso(B, B')$ square.

Nor does quotienting out by isomorphism class help us. It is *not* the case that if $B_1 \sim B$ and $B'_1 \sim B'$ then $Iso(B, B') = Iso(B_1, B'_1)$. They merely are bijective. Even if the grid worked, we still would have a dependence on B and B'.

Another approach is to hope that if $f \in Iso(B, B')$ and $f \in Iso(B_1, B'_1)$ then $Iso(B, B') = Iso(B_1, B'_1)$. I.e. that any $Iso(B, B')$ and $Iso(B_1, B'_1)$ are either disjoint or equal. If that is the case, then we can partition $Bij(S, S')$ by the set of Iso's to which each f belongs. This wouldn't respect isomorphism classes of partitions — but it at least would be a natural partition of $Bij(S, S')$, independent of any specific choice of B or B'. In fact, it would imply that our previous grid *does* consist of Iso sets (or, more precisely, sets of Iso sets since the intersection would then encompass both Iso sets). Unfortunately, this is not the case.

Suppose $f \in Iso(B, B')$ and $f \in Iso(B_1, B'_1)$. Then $Iso(B, B') = Aut(B') \circ f = f \circ Aut(B)$. However, $Iso(B_1, B'_1) = Aut(B'_1) \circ f = f \circ Aut(B'_1)$. There is no reason these should be equal. However, if $B_1 = B$, then $Iso(B, B') = Iso(B, B'_1)$ if $B' = B'_1$ and one (or both) are empty otherwise. Ditto for $Iso(B, B')$ and $Iso(B_1, B')$. Different Iso's can partly overlap, as long as both B and B' differ between them.

We therefore don't have a natural partition of $Bij(S, S')$. For any given B we have one, and for any given B' we have one, but these are not particularly useful to us.

Before moving to $Surj$, let's consider another use of Bij. Isomorphisms aren't the only bijective morphisms. We also have nonisomorphism coarsening morphisms. We could try to coarsen our isomorphism-class partition of $Par(S)$ (or $Par()$) by defining $B \sim_? B'$ iff $CMor(B, B') \neq \emptyset$. I.e., if there exists *any* bijective morphism between them. Unfortunately, this is not an equivalence relation. It is not symmetric. Coarsening defines an order relation, not an equivalence. The best we could do is try to build classes of chains, but this doesn't buy us anything.

When it comes to $Surj(S, S')$, the situation is far worse. We can't even construct a counterpart to our B-specific or B'-specific partitions of $Bij(S, S')$, let alone obtain anything resembling a natural partition of $Surj(S, S')$.

We could try to construct a counterpart to isomorphism classes in a number of ways, but none work. One approach is to define $B \sim_? B'$ iff $SMor(B, B') \neq \emptyset$. I.e., if there exists a surjective morphism. However, $\sim_?$ clearly is not an equivalence relation. It is not symmetric. For example, if $|B| > |B'|$ then $SMor(B', B) = \emptyset$. For a given B, the $SMor(B, B')$ sets indexed by B' just give us a cover of $Surj(S, S')$ but not a partition of it. Ditto for the $SMor(B, B')$ sets indexed by B for a given B'.

Our best remaining hope lies with $MapTo$. It is the surjective counterpart to Iso, and we know that a given surjective f takes a given B to at most one B' and takes exactly one B to a given B'. Suppose we define $B \sim_? B'$ iff $MapTo(B, B') \neq \emptyset$. Unfortunately, this is not an equivalence relation either. From proposition 2.30, we know that it is symmetric iff $Iso(B, B') \neq \emptyset$ or $MapTo(B, B') = MapTo(B', B) = \emptyset$. I.e., if we restrict it to the symmetric cases, we just get our isomorphism classes back. There is nothing useful here.

2.15. **Does $Surj(S, S')$ induce a counterpart of the natural bijection between $Par'(S)$ and $Par'(S')$?** Perhaps the most valuable thing to come out of our isomorphism machinery is a natural map between $Par'(S)$ and $Par'(S')$. This resulted from the fact that, for bijective f's, every $Push_f$ and $Pull_f$ induced the same quotient maps between $Par'(S)$ and $Par'(S')$.

Let's briefly review $Push_f$ and $Pull_f$ and take a closer look at the quotient maps. Recall that:

- $Push_f(B) \equiv G(f'(B))$, where G is our meet-like operation from section 1.12.
- $Pull_f(B') \equiv f^*B'$, where f^* is our pull-back from section 1.8.
- $Push_f$ is surjective. It is injective iff f is bijective. *From proposition 2.27.*
- $Pull_f$ is injective. It is surjective iff f is bijective. *From proposition 2.26.*
- $Push_f \circ Pull_f = Id_{S'}$. *From proposition 2.25.*
- $Pull_f \circ Push_f$ takes B to a coarsening of B. *From proposition 2.25.*

In the case of a bijective f, we saw that $Push_f$ and $Pull_f$ respect isomorphism classes, and we thus can define quotient maps between $Par'(S)$ and $Par'(S')$. It is these which are independent of f. Coupled with the fact that they are inverses, this provides a natural bijection between $Par'(S)$ and $Par'(S')$.

It would be nice if, for a general surjective f, $Pull_f$ and $Push_f$ also respect isomorphism classes. I.e. if $B' \sim B'_1$, then $Pull_f(B') \sim Pull_f(B'_1)$, and if $B \sim B_1$ then $Push_f(B) \sim Push_f(B_1)$. Unfortunately, neither holds.

(Example where $B' \sim B'_1$ but $Pull_f(B') \not\sim Pull_f(B'_1)$): Let $S = \{1,2,3,4,5,6\}$ and $S' = \{1,2,3,4\}$ and $B' = \{(1),(2,3,4)\}$ and $B'_1 = \{(1,2,3),(4)\}$. Then $k : (1,2,3,4) \to (4,2,3,1)$ is an isomorphism from B' to B'_1, so they are isomorphic. Consider the surjective function $f : (1,2,3,4,5,6) \to (1,2,2,3,4,4)$. $f^{-1}(B') = \{(1),(2,3,4,5,6)\}$ and $f^{-1}(B'_1) = \{(1,2,3,4),(5,6)\}$. These clearly are not isomorphic since we cannot pair up their classes by size.

(Example where $B \sim B_1$ but $Push_f(B) \not\sim Push_f(B_1)$): Let $S = \{1,2,3,4,5,6\}$ and $S' = \{1,2,3,4\}$ and $B = \{(1,2),(3,4,5,6)\}$ and $B_1 = \{(1,2,3,4),(5,6)\}$. Then $h : (1,2,3,4,5,6) \to (5,6,1,2,3,4)$ is an isomorphism from B to B_1, so they are isomorphic. Consider the surjective function $f : (1,2,3,4,5,6) \to (1,1,2,3,3,4)$. $f'(B) = ((1),(2,3,4))$ and $f'(B_1) = \{(1,2,3),(3,4)\}$. The first is a partition, so $G(f'(B)) = f'(B) = \{(1),(2,3,4)\}$. The second is not a partition, and clearly $G(f'(B_1)) = \{(1,2,3,4)\}$ is the trivial partition. These are not isomorphic.

We therefore cannot meaningfully take a quotient and define corresponding maps between $Par'(S)$ and $Par'(S')$, let alone show that they are independent of f. Another way of expressing this is as follows: in going from bijections to surjections, we move from considering isomorphic partners to considering maximally-refined and maximally-coarse partners (under a given f). We then have several (related) obstructions:

- We cannot generalize the notion of "isomorphism class" to "surjective morphism class". Even attempting to generalize it to "map-to class" gives us the usual isomorphism classes.
- Unlike isomorphic partners under a given bijective f, maximally-coarse and maximally-refined partners under a given surjective f need not be inverses. A given B need not be the maximally-coarse partner of any partition, and a given B' may be the maximally-refined partner of several partitions. Expressed in terms of $Pull_f$ and $Push_f$, $Pull_f$ need not be surjective and $Push_f$ need not be injective. Moreover, in one direction, $Pull_f$ and $Push_f$ need not be inverses. There is information loss.
- $Push_f$ and $Pull_f$ do not respect isomorphism class for non-bijective f's. We cannot define quotient maps between $Par'(S)$ and $Par'(S')$.
- Even if we could have defined such quotient maps, there is no reason to believe that they would be independent of f.

As a result, we cannot extend our quotient machinery to surjective maps in a meaningful way.